國 立 成 功 大 學
機 械 工 程 學 系
碩　士　論　文

# 針對非充裕不確定資料設計之最佳取樣與資源配置

Optimal Sampling Augmentation and Resource Allocation for Design with Inadequate Uncertainty Data

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中 華 民 國 一 百 零 一 年 一 月

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# 摘　　要

可靠度設計需有大量的樣本量測以建立不確定因素的模型，但在實際工程運用上，樣本的量測費時且昂貴。雖然，大量的量測可以提供一較為適用的設計，但設計者經常必須在有限時間內根據有限的資訊做出設計判斷。文獻中，多採用貝氏二項式推估法來進行具有樣本型態的不確定性因素的可靠度評估。然而，文獻中均假設一組不確定性參數為一個樣本，事實上，每一次量測均代表一個樣本。因此，在增加額外的樣本時，不同不確定性參數的對最終設計的貢獻差異需被納入考量。本文透過樣本組合的概念來使不確定性因素的相對重要性在增加樣本時得以被顯現。本文建立一藉由在最佳化中逐漸增加樣本來協助有效的資源分配與進行非充裕不確性資料可靠度設計的方法。為了避免量測品質不佳的樣本影響可靠度評估的準確度，因此，本研究發展一以馬可夫鏈蒙地卡羅法為基礎的樣本過濾機制來避免偏頗樣本。本研究可以在滿足可靠度目標與使用者定義的信賴區域中，透過逐漸增加少許的新樣本，並有效的配置樣本進而提供較準確的可靠度評估，獲得可接受的可靠度最佳化設計。由於信賴區域受限於樣本數量，本文定義在此樣本數量下的信賴區域的上限為信賴邊界，並將其納入可靠度最佳化的拘束條件中。額外的樣本量測在關鍵的拘束條件相關的不確定性因素上來幫助可靠度最佳化的進行。設計與額外的樣本量測會持續進行直到滿足設計者要求的目標。透過此研究方法 ，可藉由較少且較有效率的樣本量測配置進行可靠度最佳化設計。本論文以一個數學範例與一汽車懸吊系統設計演示此方法並討論結果，最後，並將汽車懸吊系統設計延伸至複雜系統來示範此設計方法。

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# ABSTRACT


Uncertainty models in reliability-based design optimization problems require a large amount of measurement data that are generally unavailable in engineering practice. Each measurement requires resources, sometimes costly. Although a comprehensive set of measurements could lead to design that is more applicable, engineers are constantly challenged to make timely design decisions with only limited information at hand. In the literature, Bayesian binomial inference techniques have been used to estimate the reliability value of a function of uncertainties with limited samples. However, existing methods assume data set as one sample for each uncertain quantity, while in reality we consider one sample as one measurement on a specific quantity. The relative contributions of uncertainties on the final optimum should be considered when adding samples.

In this thesis, we use the concept of sample combinations to reveal the relative contributions of uncertainties when adding samples. We propose a sampling augmentation process to add measurements of uncertain quantities only when they are 'important' by allocating resource more efficiently. To alleviate the impact of bad samples, biased samples that would affect the evaluation of reliability inference will be filtered via a mechanism through Markov chain Monte Carlo method. Once a desired reliability target and a user-specified confidence range are provided by the designer, a confidence bound limit that predicts the upper bound of no-failure confidence is then calculated. This confidence bound limit is then considered in a reliability-based design optimization framework as constraints. Additional measurements on critical constraints with respect to uncertainties in the form of discrete samples are necessary. Design then iterates until the desired targets are reached. In this work our method could minimize the efforts and resources without assuming distributions for uncertainties. Several examples are used to demonstrate the validity of the method in product development.

# 誌　　　　　謝

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# List of Symbols

| | |
|---|---|
| $\alpha$ | parameter of Beta distribution |
| $\beta$ | parameter of Beta distribution |
| $\pi$ | target distribution |
| $\pi_{\mathrm{AL}}$ | augmented Lagrangian penalty function |
| $\Phi_{\mathrm{Beta}}$ | cumulative density function of beta distribution |
| $a_k$ | acceptance probability of MCMC filter mechanism |
| $\mathrm{beta}(\alpha, \beta)$ | Beta distribution of parameter $\alpha$ and $\beta$ |
| $g$ | deterministic inequality constraint |
| $g_R$ | reliability inequality constraints |
| $g_B$ | Bayesian reliability inequality constraints |
| $h$ | equality constraint |
| $p$ | probability of successful outcomes |
| $p_b$ | number of bootstrap re-samplings |
| $q$ | proposal distribution |
| $r$ | number of success outcomes |
| $s_y$ | sensitivity of parameter $y$ |
| $CB$ | confidence bound |
| $CBL$ | confidence bound limit |
| $CR$ | confidence range |
| $CR_t$ | confidence range target |

| | |
|---|---|
| $N$ | number of samples |
| $N_c$ | number of sample combination |
| $R_B$ | Bayesian reliability |
| $R_k$ | expected feasible realization of one sample in Bayesian inference |
| $R_t$ | reliability target |
| $R_{MCS}$ | reliability value obtained from Monte Carlo Simulation |
| $X$ | random variable |
| $X_{\mathrm{S}}$ | random variable known for samples |
| $X_{\mathrm{U}}$ | random variable known for distribution |
| $P$ | random parameter |
| $P_{\mathrm{S}}$ | random parameter known for samples |
| $P_{\mathrm{U}}$ | random parameter known for distribution |
| Bin | Binomial process |
| $\mathbf{d}$ | deterministic design variables |
| $\mathbf{p}$ | deterministic parameters |
| $\mathbf{r}_{ij}^{i}$ | the responses to be sent to the parent subproblem |
| $\mathbf{r}_{(i+1)j}^{i+1}$ | the responses from the children subproblems |
| $\mathbf{t}_{(i+1)j}^{i}$ | the targets to the children subproblems |
| $\mathbf{t}_{ij}^{i-1}$ | the targets coming from the parent subproblem at level $i-1$ |
| $\mathbf{v}_{ij}$ | a vector of Lagrangian multiplier parameters |
| $\mathbf{w}_{ij}$ | a vector of penalty weights |
| $\mathbf{x}_{ij}$ | local variables |

| | |
|---|---|
| $\mathbf{D}$ | random design variables |
| $\mathbf{D}_{\mathrm{u}}$ | uncertain design variables known for distributions |
| $\mathbf{P}$ | random design parameters |
| $\mathbf{P}_{\mathrm{u}}$ | uncertain parameters known for distributions |
| $\mathbf{P}_{\mathrm{s}}$ | uncertain parameters known for samples |
| $\mathbf{R}$ | reliability distribution |
| MCS | Monte Carlo Simulation |
| MCMC | Markov chain Monte Carlo |
| RBDO | Reliability-Based Design Optimization |

# Chapter 1 Introduction and Motivation

## 1.1 Introduction

Reliability is one of the most critical attributes in product and process design [1]. Performances of products need to undergo various test procedures and ensure their satisfactions before they are in the hands of customers. Therefore reliability is the probability that a system satisfies a given limit state function that defines failure and success under various sources of variations and uncertainties. Figure 1.1 shows the reliability analysis procedure that quantifies the impacts of uncertainties through a limit state function. These sources of uncertainties are inevitable in product development process because they exist in the environment, in the human who uses them, in the measurements, as well as in the manufacturing processes. Due to the advancement of living quality, many customers consider reliability has a higher priority over cost when choosing a product. Therefore, in the past decades, the engineering and design community has developed various methods to improve reliability of an engineering system. In the literature, we use reliability-based design optimization (RBDO) to indicates methods that are developed to aid engineering analysis under uncertainties early in the design phase in the product development process [2].

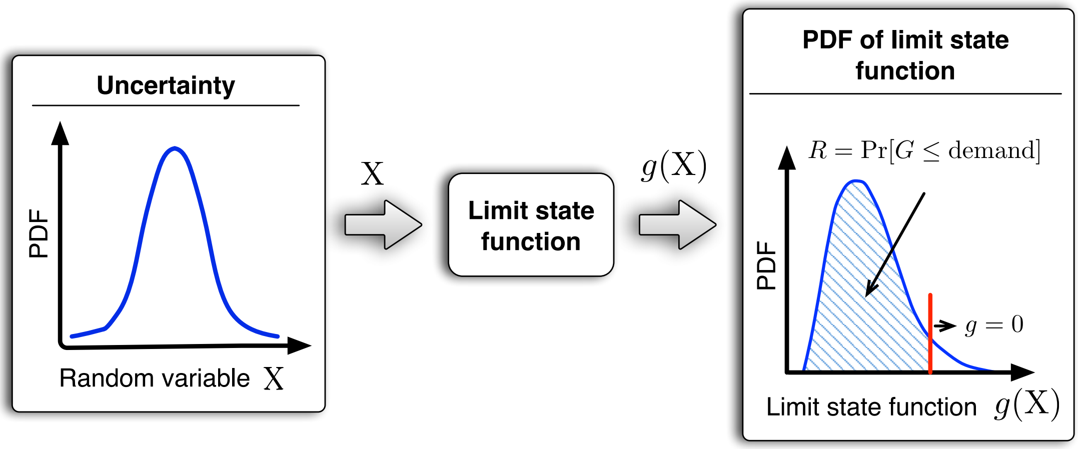


Figure 1.1: Reliability analysis procedure

In RBDO framework, a standard optimization routine is coupled with reliability analysis that requires each design candidates to be feasible yet reliable. This coupling increases the computational cost of RBDO and has been one of the most studied area in RBDO literature. Examples of these methods include: the first/second order reliability method [3–5], adaptive importance sampling [6], advance mean value [7], and its hybrid variant [8], sequential optimization and reliability assessment [9], and single-loop method [10]. Furthermore, methods for reliability assessment have been proposed to enhance numerical efficiency and stability [11–13].

The first step in reliability analysis is to obtain the models of uncertainty. Quantifying uncertainties requires a large amount uncertainty data. In the literature uncertainty can be classified into one with probability distributions and one with limited available samples. When the random property of an uncertainty can be completely known and modeled as a statistical distribution, it is defined as an "aleatory" uncertainty; whereas an uncertainty with only limited available samples is defined as an "epistemic" uncertainty. Although most RBDO research assumes the underlying distributions of all uncertainties be known, in actual engineering design, much of information regarding the uncertain quantities is only available in the form of limited samples instead of probability distributions. In fact, statistically the exact distribution of aleatory uncertainties can only be known when the one has infinite number of samples about the uncertainty. In most cases, we extract the probability distribution via inferring from samples, as Figure 1.2 shown. We can summarize that when an uncertain quantity has abundant sample

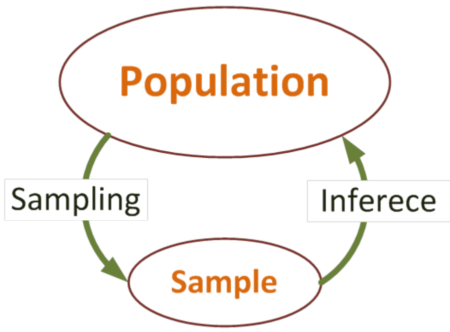


Figure 1.2: Sample Inference

measurements, we treat them as "aleatory" uncertainty with distributions inferred from the samples. However, in practical engineering application, the amount of samples is extremely restricted due to limited cost and time. As a results, the size of samples are usually not enough to infer the probability distribution of population with high confidence. We call such samples as inadequate uncertainty data.

When the available uncertainty data is inadequate, the classical probability theory [1–13] may be improper to model uncertainties. Most probability analysis and design assume that uncertainties are known for distributions can not capture the reality of engineering practice. How to use epistemic uncertainties to do reliability analysis becomes a bottleneck in engineering applications. To deal with inadequate uncertainty data, different methods have been developed in the literature. Some researchers concentrate on inferring the probability distributions of uncertainties from a small sample sets so as to make RBDO algorithms applicable [14–16]. However, inferring a probability distribution with a few samples could generates large errors and results in erroneous results in the reliability prediction [17]. Another approach focuses on reliability analysis and design optimization without inferring the probability distribution of uncertainties, such as possibility-base design optimization (PBDO) [18,19] based on possibility theory [20–25], evidence-based design optimization (EBDO) [26] based on evidence theory [27–29], and Bayesian RBDO [17,30,31] based on Bayes theory [32–35].

The possibility-based and evidence-based methods have the weakness that the uncertainties are modeled more or less based on the " expert opinions" that may be different for each expert and may even be conflicting [31]. Methods based on Bayes theory are called Bayesian approach. In this thesis, Bayesian approach is better suited to evaluate the reliability due to the advantage that (1) it provides a unified way for aleatory and epistemic uncertainty in a single framework, (2) it can conveniently update the degree of uncertainty, and (3) it is widely applied in many engineering and science fields. For example, Bayesian theory is used to estimate the multi-frequency offset to assist decision-making in a multi-objective environment [36], to access the reliability of a power network [37], to estimate the reliability of on-site lifetime measurements that are fuzzy in nature [34], to estimate the reliability of an 'inexact' small data set [35], to series systems of binomial subsystems and component [38], to the effectiveness of reliability growth testing [39], to robust tolerance control and parameter design in the manufacturing process [40], and to input uncertainty modeling [41]. Bayesian updating has been implemented using the Markov chain Monte Carlo (MCMC) simulation for structural models and reliability assessment [42]. Due to the advantages of Bayesian approach, in this thesis, Bayesian approach would be used to evaluate reliability.

## 1.2 Motivation and Objective

In engineering practice, the uncertainties are often provided as a small number of samples from historical data or actual experiments. Although several methods have been developed to tackle evaluation of reliability with limited samples, *the amount of samples is still great while considering the expense of measurements.* In existing literatures, the reliability estimation is obtained by using a set of all uncertain quantities, the redundant samples which means the samples have no significant effect on accuracy of reliability evaluation is also measured. The cost of measurement can be optimized if the measurement of redundant samples can be minimized. We would like to develop a much more efficient measurement scheme by only measuring samples when they are important. In other words, if the lack of a certain sample of an uncertainty would affect the accuracy of reliability estimation, that sample should be added.

The relative importance of different uncertainty on the reliability analysis received little attention in the literature. Without understand which sample is more critical, we are unable to cast these samples more wisely. For example, if we have three uncertain quantities, $A, B, C$. The lack of information about $A$ will decrease the accuracy of reliability estimation and have the greatest influence on optimization results compared with the other uncertainties. Therefore, with limited resources, the need to increase the sample size of $A$ has the higher priority over $B$ and $C$. In the literature, most measurement schemes suggest that an entire set of measurements on $A, B, C$ be added to existing sample when additional samples are necessary. We believe that samples should be treated differently such that their relative importance on reliability analysis can be revealed.

In this thesis, we *propose an approach to cast the minimal amount of additional samples required to achieve a specific level of accuracy in reliability analysis, and then use it in the optimal product design under inadequate uncertainty data.* In addition to the reveal the relative importance of uncertainty samples, we also emphasize on measurements that might be wrong. This biased sample measurement could alter the reliability estimation, especially for a small sample size. We would like to develop a mechanism such that these biased samples will not undermine our design process. We want to eliminate the improper samples to enhance the accuracy of reliability estimation under limited samples situation.

## 1.3 Organization of the Thesis

The reminder of this thesis is organized as shown in Figure 1.3 . We first introduce the reliability estimation of sample data via Bayesian theory and clarify meaning of samples in Chapter 2, the sample filter is also via Markov chain Monte Carlo is also introduced in Chapter 2. The proposed algorithm which doing sampling augmentation and resource allocation in optimization iteration is introduced in Chapter 3. Two single level case studies about one mathematical example and passive vehicle suspension design are demonstrated in Chapter 4. The case study about passive vehicle suspension design is also extended to complex multilevel system design in Chapter 5. Finally, conclusions and suggestions of this thesis are presented in Chapter 6.

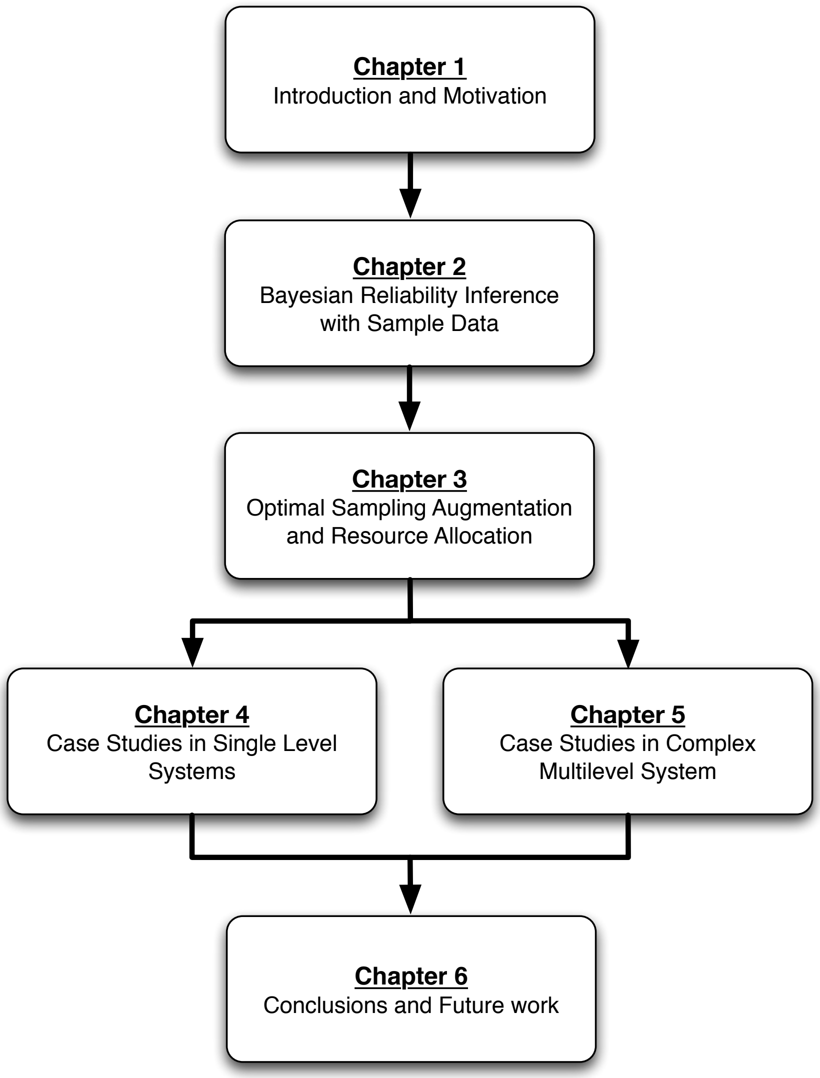


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Figure 1.3: Organization scheme

# Chapter 2 Bayesian Reliability Inference with Sample Data

The information about the underlying distributions of uncertainties are mostly assumed to be well-known in the reliability-based design optimization community. However, both aleatory and epistemic uncertainties that are common in practical engineering applications, do not always have complete information about the uncertainties. Yet we have to make appropriate design decision based on limited resources. The evaluation of reliability with inadequate uncertainty data becomes a grand challenge for designers. In recent years, Bayesian theory has been applied to tackle this challenge via inversion of probabilities. In this chapter, we will first introduce Bayesian theory in data inference in section 2.1, reliability estimation with sample data will then be discussed in section 2.2; Markov chain Monte Carlo that filters poor samples will be introduced in section 2.3.

## 2.1 Data Inference using Bayesian Theory

Bayesian theory bas been applied to infer a population by samples. Because the Bayesian theory is a new perspective of Bayes theorem via inversion of probabilities, we will first introduce Bayes theorem before talking about Bayesian inference.

### 2.1.1 Bayes Theorem

Bayes theorem is a concept of conditional probability that defines the probability of event $B$ given event $A$. Equation (2.1) shows the mathematical representation of the conditional probability of event $B$ given event $A$ happening.

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \tag{2.1}$$

$\Pr(A)$ is the probability of event $A$, $\Pr(A \cap B)$ is the probability of the joint space of both events $A$ and $B$. Similarly, the conditional probability of event $A$ on the occurrence of event $B$

is shown as Equation (2.2).

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{2.2}$$

After rearranging the fractions in the conditional probability formula, one can get

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B) \tag{2.3}$$

Equation (2.3) is known as the **multiplication rule** for probability. It restates the conditional probability relationship of an observed event given an unobservable event in a way that is useful for finding the joint probability $\Pr(A \cap B)$. If we denote that $B^c$ is the set of complement of event $B$, then

$$\Pr(A \cap B^c) = \Pr(A|B^c) \times \Pr(B^c)$$

Since $A = (A \cap B) \cup (A \cap B^c)$, the law of total probability states that the probability of event A can be calculated by summing the probability of its disjoint parts, then

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c) \tag{2.4}$$

Substituting Equation (2.4) into the definition of conditional probability, we then have

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A \cap B) + \Pr(A \cap B^c)} \tag{2.5}$$

Using the multiplication rule to find each of these joint probabilities, Bayes theorem for a single event can then be derived as :

$$\Pr(B|A) = \frac{\Pr(A|B) \times \Pr(B)}{\Pr(A|B) \times \Pr(B) + \Pr(A|B^c) \times \Pr(B^c)} = \frac{\Pr(A|B) \times \Pr(B)}{\Pr(A)} \tag{2.6}$$

Bayes theorem is a restatement of the conditional probability formula with the joint probability in the numerator being found by the multiplication rule, and the marginal probability in the denominator being found by the law of total probability followed by the multiplication rule.

Bayes theorem for a single event can be extended to general $n$ events. If an observable event $A$ follows $A = (A \cap B_1) \cup (A \cap B_2) \cdots \cup (A \cap B_i)$, the law of total probability states that the probability of an event $A$ is the sum of the probabilities of its disjoint parts. Therefore, the probability of event $A$ can be written as :

$$\Pr(A) = \sum_{j=1}^{n} \Pr(A \cap B_j) \tag{2.7}$$

Using the multiplication rule on each joint probability gives :

$$\Pr(A) = \sum_{j=1}^{n} \Pr(A|B_j) \times \Pr(B_j) \tag{2.8}$$

From definition, the conditional probability $\Pr(B_i|A)$ can also be expressed as

$$\Pr(B_i|A) = \frac{\Pr(A \cap B_i)}{\Pr(A)} \tag{2.9}$$

Using the multiplication rule in the numerator, with the law of total probability on the denominator, we can show that :

$$\Pr(B_i|A) = \frac{\Pr(A|B_i) \times \Pr(B_i)}{\sum_{j=1}^{n} \Pr(A|B_j) \times \Pr(B_j)} \tag{2.10}$$

Equation (2.10) is known as **Bayes theorem** of $n$ events which was first published in 1763 after the death of its discover, Reverend Thomas Bayes.

Note that since event $A$ and $B_i$ is with different amount of information, events $A$ and $B_i$ are not treated symmetrically. The $B_i$ are unobserved event and we do not know the outcome as a priori. The event $A$ is an observed event with probability distributions known. The probabilities $\Pr(B_i)$, called *prior* probability, are assumed known before we collecting the outcomes. The *likelihood* of the unobservable events $B_i$ is the conditional probability that A has occurred given $B_i$. Thus the likelihood of events $B_i$ is given by $\Pr(A|B_i)$. The likelihood is the weight given to each of the $B_i$ events given by the occurrence of event $A$. $\Pr(B_i|A)$ is the *posterior* probability of event $B_i$, given that event $A$ has occurred. This distribution contains the weight we attach to each of event $B_i$ after we know event $A$ has occurred. It combines our prior beliefs with the evidence given by the occurrence of event $A$.

In Bayes theorem, each of the joint probabilities (posterior probability) are found by multiplying the prior probability $\Pr(B_i)$ times the likelihood $\Pr(A|B_i)$. The only thing we need in the prior is the relative weights we give to each of possibilities. As a summary, *posterior* is proportional to the *prior* times the *likelihood*. Bayes theorem is often written in form as :

$$posterior \propto prior \times likelihood \tag{2.11}$$

A comprehensive reviews of Bayes theorem can be found in Reference [43].

### 2.1.2 Binomial Distribution

Bayes theorem states that we could update the information about the posterior by getting observations from population. As previous section shown, event $A$ is an event observed from population. Obtaining outcomes of event $A$ can be represented as sampling from an underlying distribution. Furthermore, sampling from a underlying distribution can be referred to as random sampling from a very large population that follows the binomial distribution same as coin tossing. In order to realize, in what follows, the case with coin tossing is briefly described, followed by random sapling from a large population. We intend to show the similarities between coin tossing and random sampling.

#### Coin tossing

A coin is tossed $N$ times and count the number of heads occurring. The outcome of one toss is independent of the outcome of previous toss. The probability of getting heads is the same value for all tosses due to uses of the same coin. The probability of getting head is denoted as $p$. Getting head is referred as " success."

#### Random sampling from a very large population

We draw a set of samples with size $N$. Assuming all draws are taken under the same duplicate conditions. Some items in the population have certain attribute $s$. We count the number of the items having attribute as in coin tossing. The outcome of any draw is independent of previous outcomes. The probability of having certain attribute $s$ is denoted as $p$. Having the attribute $s$ is referred as " success."

Both random sampling from a very large population and coin tossing have properties that meet the characteristics of a binomial distributions. There are several properties in common. These properties are characteristics of binomial distribution.

These characteristics include:

- There are $N$ *independent* trials with outcomes being either " success" or " failure".

- The probability of "success" is constant over all the trials. Let $p$ be the probability of " success."
- $r$ is the number of "successes" that occurred in the $N$ trials. $r$ can take on integer values $0, 1, \ldots, N$.

Sampling from an underlying distribution can be referred to as a binomial distribution. The binomial random variable $r$ given the parameter value $p$ is expressed as probability density function (pdf) $f$

$$f(r|p) = \begin{pmatrix} N \\ r \end{pmatrix} p^r (1-p)^{N-r} \tag{2.12}$$

for $r = 0, 1, \cdots, N$ where

$$\begin{pmatrix} N \\ r \end{pmatrix} = \frac{N!}{r! \times (N-r)!}$$

We can use the binomial random variable to calculate the likelihood of "success" event in the following section.

### 2.1.3 Bayesian Binomial Inference

In a large population, we denote $p$ as a proportion of the population with some attributes. If we want to know the probability of having some attribute in population, we need to take a random sample from the population and make inference of population. Bayesian inference is new perspective of Bayes theorem. Bayesian inference can be applied to different distributions. Since sampling from an underlying distribution can be modeled as a binomial random variable, we will only introduce the Bayesian binomial inference. The comprehensive reviews of Bayesian Statistics can be found in [43–47].

In this section, we briefly summarize the Bayesian binomial inference and its underlying assumptions. Let an event with outcomes modeled as a binomial process. Given $N$ trials, the probability of having $r$ success outcomes can be presented by $r \sim \text{Bin}(N, p)$ where Bin() is the binomial process and $p$ is the probability of successful events. The conditional probability density function $f$ of $r$ given $p$ follows binomial random variable function (express in probability density function) as Equation (2.13)

$$f(r|p) = \begin{pmatrix} N \\ r \end{pmatrix} p^r(1-p)^{N-r} \text{ for } y = 1, \ldots, N \tag{2.13}$$

Since we do not know the probability of successful events $p$, we let $r$ be fixed at the number of success outcome we observed and $p$ vary over its possible values, the likelihood function is then :

$$f(r|p) = \begin{pmatrix} N \\ r \end{pmatrix} p^r(1-p)^{N-r} \text{ for } 0 \leq p \leq 1 \tag{2.14}$$

We see that we are looking at the same relationship as the distribution of the observation $r$ given the parameter $p$ in Equation (2.13) and (2.14). But the subject of the Equation (2.14) has changed to the parameter for the observation held at the value that actually occurred.

From Bayes theorem, *posterior is proportional to the prior times the likelihood*. Therefore :

$$f(p|r) \propto f(p) \times f(r|p) \tag{2.15}$$

Equation (2.15) only gives the shape of the posterior distribution. To get actual posterior, Bayes theorem states that the distribution of $p$ can be obtained using conditional probability concept by divided a normalized factor $\int_0^1 f(p) \times f(r|p)$ to make sure the posterior is a probability distribution, meaning the area under posterior integrates to 1.

$$f(p|r) = \frac{f(p) \times f(r|p)}{\int_0^1 f(p) \times f(r|p)dp} \tag{2.16}$$

where $f(p)$ is the prior distribution of $p$, $f(p|r)$ is the posterior distribution of $p$ with $r$ success, and $f(r|p)$ is the likelihood of $r$ given $p$.

If we want to use Bayes theorem from Equation (2.16), we need a prior distribution $f(p)$ about the possible value of the parameter $p$ before getting the data. It is important that the prior can not construct from the data. In order words, the prior need to be independent of the likelihood. This means that the observed data must not have any influence on the choice of prior. Following, we will look at some possible priors.

### Using a Uniform Prior

If we don't have any idea about the prior distribution $f(p)$, or may want to be an objective as possible and not put personal belief into the inference, we should choose a prior that does not favor any value over another. In this case, the *uniform* prior that gives equal weight to all possible values should be used. The formulation about a uniform prior is

$$f(p) = 1 \text{ for } 0 \leq p \leq 1 \tag{2.17}$$

With a uniform prior, the posterior distribution is proportional to the likelihood as :

$$f(p|r) = \begin{pmatrix} N \\ r \end{pmatrix} p^r (1-p)^{N-r} \text{ for } 0 \leq p \leq 1 \tag{2.18}$$

Equation (2.18 ) shows that the posterior follows a beta distribution beta$(\alpha, \beta)$ where $\alpha = r+1$ and $\beta = N - r + 1$. Therefore, the posterior distribution of $p$ given $r$ is easily obtained. We didn't to do any integration but only to look at the exponents of $p$ and $(1-p)$.

### Using a Beta Prior

Suppose a beta$(\alpha, \beta)$ prior density is used for $p$ :

$$f(p : \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{(\alpha-1)}(1-p)^{(\beta-1)} \text{ for } 0 \leq p \leq 1 \tag{2.19}$$

The posterior is proportional to the prior times the likelihood as :

$$f(p|r) \propto \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{(\alpha-1)}(1-p)^{(\beta-1)} \times \begin{pmatrix} N \\ r \end{pmatrix} p^r (1-p)^{N-r} \tag{2.20}$$

Since the either the prior and the likelihood will neither be affected by multiplying by a constant, we could ignoring the constant that don't depend on the parameters $r, p$ in Equation (2.20), This gives :

$$f(p|r) \propto p^{(\alpha+r-1)}(1-p)^{(\beta+N-r-1)} \text{ for } 0 \leq p \leq 1 \tag{2.21}$$

Recognizing Equation (2.21) with beta distribution with parameter $\alpha' = \alpha+r$ and $\beta' = \beta+N-r$. We can discover that we only need to add the number of successes to $\alpha$ and add the number of failures to $\beta$ of observations :

$$f(p|r) = \frac{\Gamma(\alpha+\beta+N)}{\Gamma(\alpha+r)\Gamma(N-r+\beta)} p^{(r+\alpha-1)}(1-p)^{(N-r+\beta-1)} \tag{2.22}$$

for $0 \leq p \leq 1$. The most important merit is that the posterior density can be obtained without going through the integration. There is one more thing to be noted that the uniform prior is a special case of the beta distribution where is beta$(1, 1)$.

**Conjugate Family of Priors for Binomial Observation is the Beta Family**

Uniform and Beta priors enable us to calculate the posterior distribution by simply adding the exponents of $p$ and $(1-p)$, respectively. Therefore, for a beta prior distribution and a binomial likelihood, we can get a beta posterior by the simple rule "add successes to $\alpha$, add failures to $\beta$." There is a big advantage that all we have to do is use the observations to update the parameter of the conjugate family prior to obtain the conjugate family posterior.

## 2.2 Reliability Estimation with Sample Data

Design optimization with both aleatory and epistemic uncertainties should take reliability into consideration. Reliability is the probability of acceptance of a quality function $q$. However, the reliability estimation can not be evaluated by the same concept with inadequate uncertainty data. The evaluation of reliability without adequate uncertainty data can be obtained by the concept from Bayesian inference. Therefore, in this section, we will first introduce the Bayesian inference of reliability, then define properties about confidence level of reliability estimation, then clarify about the definition of " sample ", and then give a reliability estimation example.

### 2.2.1 Bayesian Inference of Constraint Reliability Values

Let $N$ be the number of samples, $p$ be the probability of successful outcomes, and $r$ be the number of successful outcomes. Before we measure samples, we have no idea about the prior of $p$; therefore, a uniform prior for $p$, $p \sim$ beta$(\alpha = 1, \beta = 1)$, and a binomial likelihood $f(r|p)$ are used. Based on Equation (2.16), the posterior $f(p|r)$ follows a beta distribution with parameters $\alpha = r + 1$ and $\beta = (N - r) + 1$

$$f(p|r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{(\alpha-1)}(1-p)^{(\beta-1)} \tag{2.23}$$

The updated distribution for $p$ is then beta$(r+1,(N-r)+1)$. Eq.(2.16) can be used iteratively to update $p$ with the added information of $N$ and $r$.

Bayesian binomial inference in updating probability distributions can be used to calculate the probability of a function with discrete samples. Let $X_{\mathrm{S}}$ and $P_{\mathrm{S}}$ known for samples and $X_{\mathrm{U}}$ and $P_{\mathrm{U}}$ known for distributions of the random variable $X$ and parameter $P$, respectively. To obtain the reliability distribution $\mathbf{R}$, reliability inference must be performed at every sample point while considering uncertainties known for distributions. The probability of a constraint $g(X,P)$ being feasible given the $k$th sample set is obtained at different sample points for uncertainties known for distributions as Figure 2.1 and Equation (2.24).

$$R_k = \Pr[g(X_{\mathrm{U}}, P_{\mathrm{U}})|_{(X_{\mathrm{S}},P_{\mathrm{S}})_k} \leq 0] \tag{2.24}$$

Since a constraint being feasible is generally referred to as 'reliable', we use $R_k$ instead of $p_k$

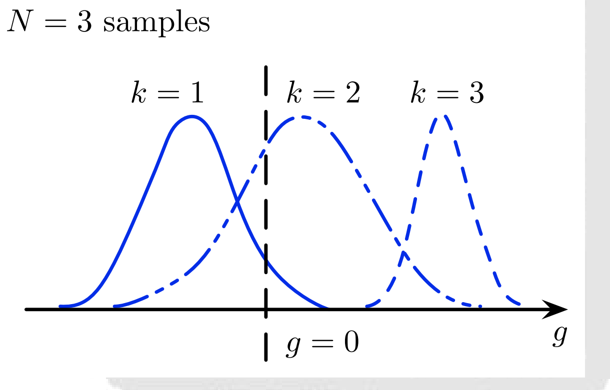



Figure 2.1: Feasible-infeasible realization of a $(X_{\mathrm{S}}, P_{\mathrm{S}})$ sample given distributions of $(X_{\mathrm{U}}, P_{\mathrm{U}})$

in the remaining of the text. The probability $R_k$ is the expected feasible realization of one sample. Therefore, the expected total number of successful events $E[r]$ with $N$ samples of feasible realizations is the sum of the probabilities of all samples :

$$E[r] = \sum_{k=1}^{N} R_k \tag{2.25}$$

Equation (2.25) is valid for both when only samples are available and when there is a mix of samples and distributions. Using $E[r]$ as the number of successful outcomes, the resulting posterior reliability distribution $\mathbf{R}$ is followed a beta distribution with parameters $\alpha = E[r]+1$ and $\beta = N - E[r] + 1$ from Bayesian binomial inference as :

$$\mathbf{R} \sim \text{beta}(E[r]+1,(N-E[r])+1) \tag{2.26}$$

### 2.2.2 Confidence Range and Confidence Bound

With the availability of the distribution of $\mathbf{R}$ with different $k$, we define confidence range and confidence bound of the reliability calculation in this work. The confidence range ($CR$) is the likelihood probability of the estimation of $\mathbf{R}$ being greater than a target $R_{\mathrm{t}}$ as:

$$CR = [\Pr(\mathbf{R} > R_{\mathrm{t}})] = 1 - \int_{x=0}^{R_t} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = 1 - \Phi_{\mathrm{Beta}}(R_{\mathrm{t}}, \alpha, \beta) \quad (2.27)$$

The confidence bound ($CB$) is the right-most distribution of all infinite number of possible beta distribution which capturing the highest confidence level given $N$ and $R_{\mathrm{t}}$. In other words, $CB$ is the upper bound of $CR$. The upper bound of the beta distribution occur when $E[r] = N$ so the parameters of beta distribution is $\alpha = N+1$ and $\beta = 1$. From the definition, the confidence bound can be obtained as

$$CB = \max[\Pr(\mathbf{R} > R_{\mathrm{t}})] = 1 - \Phi_{\mathrm{Beta}}(R_{\mathrm{t}}, N+1, 1) \quad (2.28)$$

This equation can be simplified by substituting the $\alpha = N+1$ and $\beta = 1$ into Equation (2.27)

$$CB = 1 - \int_{x=0}^{R_t} \frac{\Gamma((N+1)+1)}{\Gamma(N+1)\Gamma(1)} x^N dx \quad (2.29)$$

We know $\Gamma(1) = 1$ and $\Gamma((N+1)+1)/\Gamma(N+1) = N+1$,the constant $\frac{\Gamma((N+1)+1)}{\Gamma(N+1)\Gamma(1)}$ reduces to

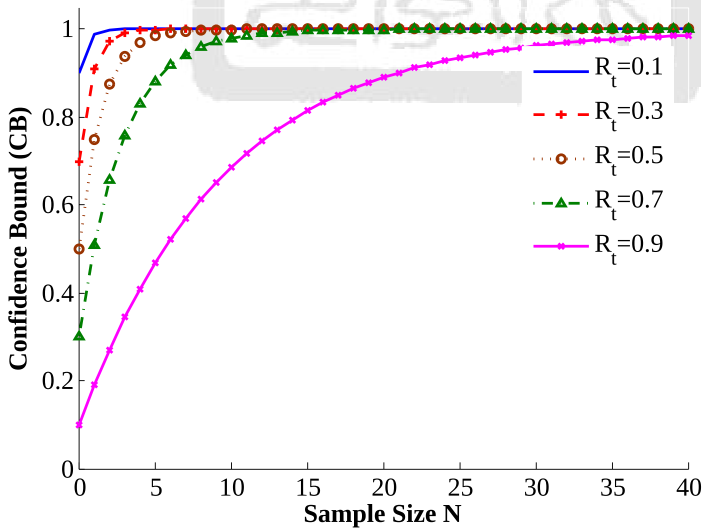



Figure 2.2: The $N$-$R_t$-$CB$ diagram

just $N+1$. The Equation (2.29) becomes

$$CB = 1 - (N+1)\int_{x=0}^{R_t} x^N dx = 1 - R_t^{N+1} \quad (2.30)$$

This simple equation link the three quantities of interest $N$, $R_t$ and $CB$ together. Equation (2.30) shows that the confidence bound of reliability target is a function of the sample size ($N$). Figure 2.2 shows the relationship between the sample size and confidence bound with different reliability target. We observe in this diagram that increasing $N$ or decreasing $R_t$ will increase $CB$. In order to provide higher confidence range, the increment of samples is necessary.

### 2.2.3 Clarification of " Sample"

Reliability information with inadequate uncertainty data follows a beta distribution. As the number of samples $N$ increases, the estimation of reliability becomes more precise. When $N \to \infty$, the distribution converges to a deterministic value with $R$ being known exactly. The case with infinite numbers of coincide with the assumption that the pdfs of all uncertainties are known.

Methods in current literatures [17,30,35] assume data set include one sample as a set of all uncertain quantities. This assumption may not reveal the true practice in the industry where one sample means one measurements. For example, in vehicle suspension design, there

Table 2.1: Parameters in the form of samples in vehicle suspension design

| Parameters | Samples |
|---|---|
| Road irregularity, $A$ | $A_1, A_2$ |
| Oil density, $\rho$ | $\rho_1, \rho_2$ |
| Oil dynamic viscosity, $nu$ | $\nu_1, \nu_2$ |
| Sprung mass, $M$ | $M_1, M_2$ |
| Unsprung mass, $m$ | $m_1, m_2$ |

are several parameters in the form of limited samples, such as road irregularity, oil density, and oil dynamic viscosity in damper, sprung and unsprung mass of vehicle. If we have two measurements for each parameter as shown in Table 2.1, existing methods treat them as two samples. However, these five parameters cannot be measured at same time using the same instrument. We cannot decide which sample of a parameter should be matched to which of another parameter to be one sample of a set of uncertainties. The relative contributions of uncertainties should be considered. The concept of sample combination reveals the all possible situation that each sample be matched. In this situation, we say that there are 10 samples and $2^5 = 32$ sample combinations. We denote the number of sample combination as $N_c$ in the remaining the text and use the concept of sample combinations instead of number of sets of all uncertainties as one sample.

### 2.2.4 Reliability Estimation Example

In this section, we will demonstrate Bayesian reliability inference using a mathematical example. Let $G(\mathbf{P}_1, \mathbf{P}_2) = 1 - 80/(\mathbf{P}_1^2 + 8\mathbf{P}_2 - 6.5) \leq 0$ be an inequality constraint with two epistemic random parameters $\mathbf{P}_1$, and $\mathbf{P}_2$. The underlying distributions of $\mathbf{P}_1$ and $\mathbf{P}_2$ are of $\mathbf{P}_1 \sim N(-8.2, 0.08^2)$ , $\mathbf{P}_2 \sim N(2.2, 0.02^2)$, but we assume this is not know to the engineers. Some samples draw for $\mathbf{P}_1$ and $\mathbf{P}_2$ are shown in Table 2.2. The underlying distributions of both parameters were unknown and therefore samples in Table 2.2 are the only information. When no pdfs are given, each probability $R_k$ in Equation (2.24) becomes an indicator function where $I_k = 1$ if $G((\mathbf{P}_1, \mathbf{P}_2)_k) \leq 0$, and $I_k = 0$ otherwise.

In the first case, lets' assume we have 5 initial samples ( the first line in Table 2.2 for $\mathbf{P}_1, \mathbf{P}_2$). The number of trial ($N$) is not 5, instead, we use the concept " Sample Combination" defined as number of combinations ($N_c$). $N_c = 5^2 = 25$.

Table 2.2: Samples of $\mathbf{P}_1, \mathbf{P}_2$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mathbf{P}_1$ | initial samples | -8.191 | -8.199 | -8.196 | -8.214 | -8.213 |
| | additional measurements | -8.201 | -8.191 | -8.180 | -8.207 | -8.196 |
| | | -8.200 | -8.190 | -8.198 | -8.180 | -8.190 |
| $\mathbf{P}_2$ | initial samples | 2.228 | 2.187 | 2.225 | 2.194 | 2.1881 |
| | additional measurements | 2.210 | 2.136 | 2.184 | 2.232 | 2.208 |
| | | 2.211 | 2.177 | 2.190 | 2.191 | 2.182 |

By carrying out the probability analysis for all 25 combinations of samples, we obtain the indicator function $I_k$ of each sample combination, then the expected total number of successful events $E[r]$ in Equation (2.25) with $N_c$ sample combinations of feasible realizations is the sum of the probabilities of all sample combinations $E[r] = \sum_{k=1}^{N_c} I_k$. Therefore, summing of probabilities of all sample combinations, we get $E[r] = \sum_{k=1}^{N_c} I_k = 21$, then using Equation (2.26), the reliability can then be modeled with the beta distribution $\mathbf{R} \sim \text{beta}(E[r]+1, (N_c - E[r])+1) = \text{beta}(22, 5)$. The reliability target $R_t = 0.9$, we can calculate the maximum probability these combination can exceed the reliability target as confidence bound $CB = \max[\Pr(\mathbf{R} > R_t)] = 1 - \Phi_{Beta}(0.9, 26, 1) = 0.9354$ that means these combinations' maximum confidence range, then the confidence range becomes $CR = [\Pr(\mathbf{R} > R_t)] = 1 - \Phi_{Beta}(0.9, 22, 5) = 0.1118$.

To show how confidence range changes with more samples, we demonstrate the second case when sample size of $\mathbf{P}_1$ and $\mathbf{P}_2$ both are 15, $E[r] = \sum_{k=1}^{N_c} I_k = 209$. The reliability distribution becomes $\mathbf{R} \sim \text{beta}(E[r]+1, (N_c - E[r])+1) = \text{beta}(210, 17)$ and the confidence range becomes $CR = [\Pr(\mathbf{R} > R_t)] = 1 - \Phi_{Beta}(0.9, 210, 17) = 0.9166$. Case1 is the reliability distribution of 25 combinations and case 2 is of 225 combinations. As shown in Figure 2.3, we assert that scenario 2 has better confidence range than the scenario 1 and that the more sample combinations, the more precious estimate of the reliability distribution.

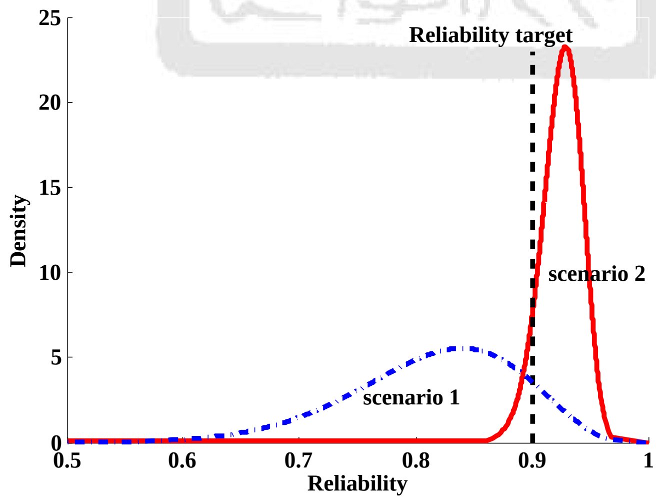



Figure 2.3: Reliability distribution with different sample combination of reliability estimate example

## 2.3 Sample Data Filter via Markov Chair Monte Carlo

Measurement could go wrong. The quality of sample measurement would affect the accuracy of reliability estimation. In order to avoid higher biased samples, a mechanism to filter out samples is necessary. In this research we use the **Markov Chain Monte Carlo(MCMC)** method as the filter mechanism.

### 2.3.1 Backgrounds on Markov Chain

Let $X_n$ denote the value of a random variable at time $n$, and let the state space refer to the range of possible $X$ value. A random variable follows a Markov process if its transition probabilities between different values in the state space depends only on the random variable's current state. That is, we suppose that

$$\Pr\{X_{n+1} = j | X_0 = i_0, .X_1 = i_1, .., X_n = i\} = \Pr\{X_{n+1} = j | X_n = i\} \tag{2.31}$$

In order words, the current state of the random variable is the only information about the past to predict the future of a Markov random variable. The other information would not affect the transition probability. Therefore, a Markov chain process can be defined by its **transition probabilities** (or the **transition kernel**) $P_{ij}$, which represents the probability that the process in state $i$ will transit into the state $j$ as Equation (2.32)

$$P_{ij} = \Pr\{i \to j\} = \Pr\{X_{n+1} = j | X_n = i\} \tag{2.32}$$

Since the probabilities are nonnegative and the process must make a transition into some state, we have :

$$\begin{gathered} P_{ij} \geq 0, i, j \geq 0; \\ \sum_{j=0}^{\infty} P_{ij} = 1, i = 0, 1, \ldots \end{gathered} \tag{2.33}$$

Let $\mathbf{P}$ denote the matrix of one-step transition probabilities$P_{ij}$, so that

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{2.34}$$

The one-step transition probabilities $P_{ij}$ has already been defined as Equation (2.32). Now, we define the $n$-step transition probabilities $P_{ij}^n$ to be the probability that a process in state $i$ will be in state $j$ after $n$ additional transitions. That is,

$$P_{ij}^n = \Pr\{X_{n+k} = j | X_k = i\},\ n \geq 0, i, j \geq 0 \tag{2.35}$$

The *Chapman-Kolmogorov equations* provide a method for computing these $n$-step transition probabilities. These equations are as Equation (2.36) shown

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \text{for all } n, m \geq 0,\ \text{all } i, j \tag{2.36}$$

and are most easily understood by noting that $P_{ik}^n P_{kj}^m$ represents the probability that starting in $i$ the process will go to state $j$ in $n+m$ transitions through a path which takes it into state $k$ at the $n$th transition. Hence, summing over all intermediate states $k$ yields the probability that the process will be in state $j$ after $n+m$ transitions. Formally, we have

$$\begin{aligned} P_{ij}^{n+m} &= \Pr\{x_{n+m} = j | X_0 = i\} \\ &= \sum_{k=0}^{\infty} \Pr\{X_{n+m} = j, X_n = k | X_0 = i\} \\ &= \sum_{k=0}^{\infty} \Pr\{X_{n+m} = j | X_n = k, X_0 = i\} \Pr\{X_n = k | X_0 = i\} \\ &= \sum_{k=0}^{\infty} P_{kj}^m P_{ik}^n \end{aligned} \tag{2.37}$$

Let $\mathbf{P}^{(n)}$ denote the matrix of $n$-step transition probabilities $P_{ij}^n$. Equation (2.36) then states :

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \cdot \mathbf{P}^{(m)} \tag{2.38}$$

where the dot represents matrix multiplication. Hence, by induction

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^{n} \tag{2.39}$$

That means that the $n$-step transition matrix may be obtained by multiplying the matrix $\mathbf{P}$ by itself $n$ times. If the limit of the Markov chain $\pi_j$ in Equation (2.40) exists and is independent of $i$,

$$\pi_j = \lim_{n\to\infty} P_{ij}^{n}, j \geq 0 \tag{2.40}$$

then $\pi_j$ is the unique nonnegative solution of :

$$\begin{aligned} \pi_j &= \sum_{i=0}^{\infty} \pi_i P_{ij}, j \geq 0, \\ \sum_{j=0}^{\infty} \pi_j &= 1 \end{aligned} \tag{2.41}$$

When the initial state is chosen according to the probabilities $\pi_j, j \geq 0$, the probability of being in state $j$ at any time $n$ equals to $\pi_j$. Therefore, $\pi_j, j \geq 0$, is called *stationary* probabilities. Mathematically, we can state that if

$$\Pr\{X_0 = j\} = \pi_j, j \geq 0 \tag{2.42}$$

then

$$\Pr\{X_n = j\} = \pi_j \text{ for all } n, j \geq 0 \tag{2.43}$$

Equations (2.42) and (2.43) can be proved by induction, for if we suppose it true for $n$, then the stationary probability can be derived by conditioning on the state at time $n$. That is,

$$\begin{aligned} \Pr\{X_{n+1} = j\} &= \sum_i \Pr\{X_{n+1} = j | X_n = i\} \Pr\{X_n = i\} \\ &= \sum_i P_{ij}\pi_i \text{ by the induction hypothesis} \\ &= \pi_j \text{ by Equation (2.41)} \end{aligned} \tag{2.44}$$

It can be shown that $\pi_j$, the limiting probability that the process will be in state $j$ at time $n$, also equals the long-run proportion for time that the process will be in state $j$. Comprehensive reviews of Markov chain can be found in [48–50].

### 2.3.2 Markov Chain Monte Carlo by Metropolis-Hasting Algorithm

A Markov chain that generates samples randomly from previous samples is called a Markov chain Monte Carlo (MCMC). MCMC ensure that the transition probabilities between sample values are only function of the most recent sample values. The application of MCMC methods on different fields has shown great impact in recent decades. Statisticians, physicists and engineers attempt to compute complex integrals by expressing them as expectations for some distribution and then estimate this expectation by drawing samples from that distribution. To solve this problem are the roots of MCMC methods. In this thesis, we randomly draw samples from population, we want to construct a mechanism to filter out samples. The characteristic of Markov chain that the future state is only related to current state let us do not to considerate the past samples data. Therefore, MCMC is method that generates samples which is quite close the situation in this thesis. Therefore, we use the MCMC to construct the filter mechanism. In different MCMC methods, a considerable amount of attention is being devoted to the Metropolis-Hastings(M-H) algorithm, which was developed by Metropolis, Rosenbluth, Teller(1953), and subsequently generalized by Hastings(1970). Following, we will introduce the MCMC by Metropolis-Hasting algorithm.

**Metropolis-Hasting algorithm**, can be used to generate a time reversible Markov chain whose stationary probabilities are $\pi(j), j = 1, 2, \ldots$. Suppose our goal is to samples from target distribution $\pi$ and very difficult to compute. Then we start with Metropolis-Hasting algorithm.

To begin, let $\mathbf{Q}$ be any specified Markov transition probability matrix on the integers, with $q(i,j)$ representing the row $i$ column $j$ element of $\mathbf{Q}$. Now define a Markov chain $\{X_n, n \geq 0\}$ as follows. When $X_n = i$, generate a random variable $Y$ such that $\Pr\{Y = j\} = q(i,j), j = 1, 2, \ldots$. If $Y = j$, then set $X_{n+1}$ equal to to $j$ with probability $a_k(i,j)$, which is referred as the *probability of move*, and if the move is not made, the process again returns $i$ as a value from the stationary probability with probability $1 - a_k(i,j)$. Under these condition, the sequence of states constitutes a Markov chain with transition probability $P_{i,j}$ as

$$\begin{aligned} P_{i,j} &= q(i,j)a_k(i,j), \text{if } j \neq i \\ P_{i,i} &= q(i,i) + \sum_{k \neq i} q(i,k)(1 - a_k(i,k)) \end{aligned} \tag{2.45}$$

where $q(i,j)$ is referred to as the *proposal* or *candidate-generating distribution*, represents that when a process is at the point $i$, the density generates a value $y$ from $q(i,j)$.

This Markov chain $X_n$ will be time reversible and have stationary probabilities $\pi(j)$ if

$$\begin{aligned} \pi(i)P_{i,j} &= \pi(j)P_{j,i} \text{ if } j \neq i \\ \pi(i)q(i,j)a_k(i,j) &= \pi(j)q(j,i)a_k(j,i) \end{aligned} \tag{2.46}$$

To show the reversibility for the Markov chain, let us set

$$a_k(i,j) = \min\left(\frac{\pi(j)q(j,i)}{\pi(i)q(i,j)}, 1\right) \tag{2.47}$$

then Equation (2.46) is also satisfied. If

$$a_k(i,j) = \frac{\pi(j)q(j,i)}{\pi(i)q(i,j)} \tag{2.48}$$

then $a_k(j,i) = 1$ and Equation (2.46) follows, if $a_k(i,j) = 1$ then :

$$a_k(j,i) = \frac{\pi(i)q(i,j)}{\pi(j)q(j,i)} \tag{2.49}$$

The probabilities $a_k(i,j)$ and $a_k(j,i)$ are thus introduced to ensure that $\pi(j)$ satisfies the reversibility. Thus we have shown that in order for $\pi(j)$ to be reversible, the probability of move must be set to

$$\begin{aligned} a_k(i,j) &= \min\left(\frac{\pi(j)q(j,i)}{\pi(i)q(i,j)}, 1\right), \text{ if } \pi(i)q(i,j) > 0 \\ &= 1, \text{ otherwise.} \end{aligned} \tag{2.50}$$

We now summarize the Metropolis-Hasting algorithm in algorithmic form initialized with arbitrary value $s_0$ and suppose that our goal is to draw samples from target distribution $\pi$ :

1. Repeat for $j = 1, 2, \ldots, N$.

2. Using current $s_j$ value, draw a candidate point $s^*$ from some proposal distribution $q(s_1, s_2)$, which is the probability of returning a value of $s_2$ given a previous value of $s_1$. The proposal distribution q is essentially arbitrary provided it can move around the entire space.

3. Generate $u$ from $U(0,1)$.

4. Calculate the probability of move $a_k$, which is also called *acceptance probability*.

$$a_k(s_j, s^*) = \min\left(\frac{\pi(s^*)q(s^*, s_j)}{\pi(s_j)q(s_j, s^*)}, 1\right)$$

5. If $u \leq a_k(s_j, s^*)$, set $s_{j+1} = s^*$, go to step 7.

6. Else, set $s_{j+1} = s_j$, go to step 7.

7. Return the values $\{s_1, s_2, \ldots, s_N\}$.

A detailed review about Metropolis-Hasting algorithm is given by Chib and Greenberg [51].

### 2.3.3 MCMC Modification with Bootstrap

Section 2.3.2 describe the Metropolis-Hasting algorithm to draw samples from a known target distribution $\pi$. However, in this work, we attempt to draw actual samples from population which the underlying distribution is mostly unknown. The underlying distribution is the target distribution $\pi$ in Metropolis-Hasting algorithm. To apply Metropolis-Hasting algorithm, we need to know the target distribution $\pi$ and proposal distribution $q$. Therefore, we must construct these two distributions. The target distribution will be estimated directly by existing samples. The proposal distribution must be different from target distribution. Therefore, we use Bootstrap concept to generate a different distribution as proposal distribution. The details of the Bootstrap concept and two modifications of MCMC for sample filtering will be introduced in this section.

**Bootstrap concept**

Based on Metropolis-Hasting algorithm the more similar the proposal distribution and target distribution, the higher convergent rate of Metropolis-Hasting algorithm. The bootstrap method can provide an efficient way of estimating the distribution using the re-sampling technique [52,53]. Therefore, the bootstrap method is used to estimate the statistical parameter of the proposal distribution $q$. Figure 2.4 illustrate the procedure of the bootstrap method. The idea of bootstrap method is to generate many set of bootstrap sample by re-sampling with

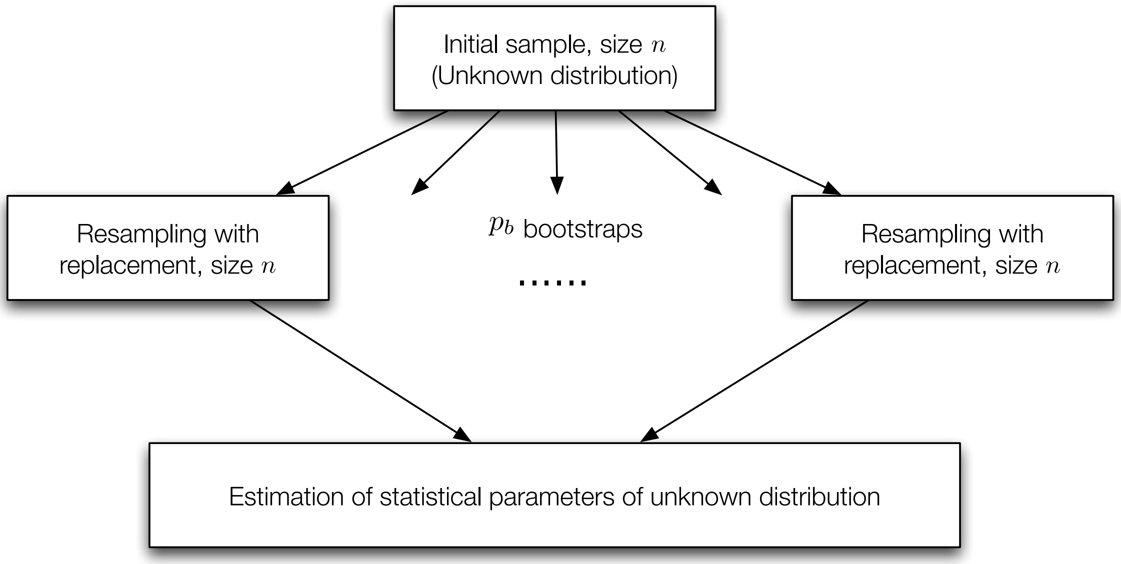



Figure 2.4: The procedure of bootstrap

replacement from original samples. Let the size of the initial samples be $n$ and the number of bootstrap re-samplings be $p_b$. Each resampling procedure selects $n$ samples with replacement of $n$ original data. The estimate statistical parameters of unknown distribution by $p_b$ bootstraps.

If the acceptance probability always be one, we can not use the Metropolis-Hasting algorithm to construct sample data filter. Therefore, we use Bootstrap method to generate a proposal distribution which different from directly estimate from original samples to avoid the acceptance probability in Equation (2.47) of MCMC always be one.

### MCMC of Accepting the Current Sample

The accuracy of a reliability estimation would be affected by biased samples. Therefore, we need to construct a filter mechanism to ensure a fair judgment. The acceptance probability of Metropolis-Hasting algorithm in MCMC is referred to as the probability of the move of a new sample.

In this work, we want to use the acceptance probability as a judgement about biased samples. If we want to use the concept of acceptance probability, the target distribution and generating distribution must be known. We obtain the target distribution by use the statistic toolbox in commercial tool Matlab of existing samples. And the proposal distribution is obtained by re-sampling technique of Bootstrap method.

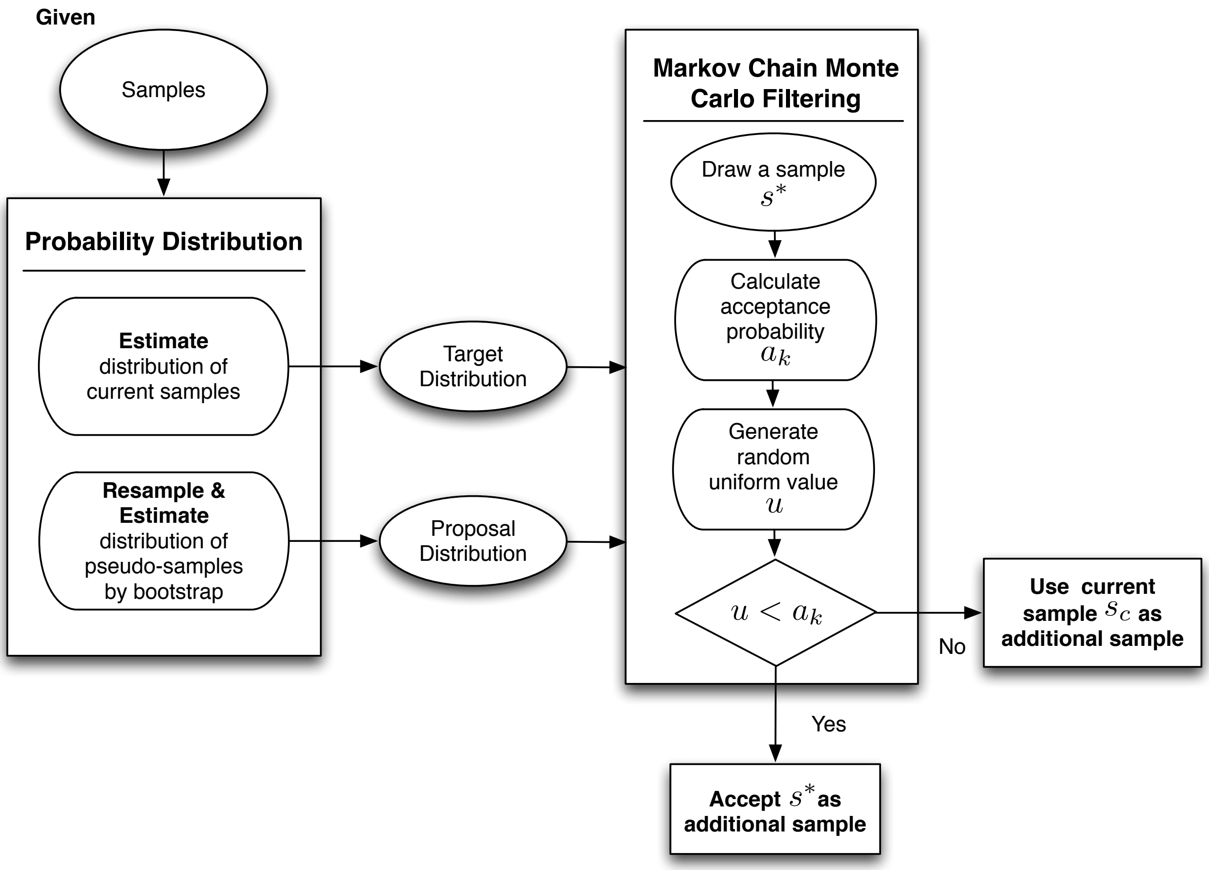



Figure 2.5: The procedure of MCMC of accepting the current sample

Figure 2.5 illustrates the procedure about the sample data filter via MCMC of accepting the current sample. We summarize the procedure in algorithmic form as following:

1. Estimate statistical parameters of the underlying distribution by directly using existing samples as target distribution $\pi$.

2. Resample and estimate the statistical parameters by bootstrap method as the proposal distribution $q$.

3. Draw a sample $s^*$ from population, calculate the acceptance probability $a_k$ via the current

sample $s_c$.

$$a_k(s_c, s^*) = \min\left(\frac{\pi(s^*)q(s^*, s_c)}{\pi(s_c)q(s_c, s^*)}, 1\right)$$

4. Generate $u$ from $U(0,1)$.

5. If $u \le a_k(s_c, s^*)$, then accept sample $s^*$ as additional sample.

6. Else, set current sample $s_c$ as additional sample.

### MCMC of Accepting An Additional Sample

The original concept of Metropolis-Hasting use the current sample as the new sample when the rejection occurring. We are afraid that if the current sample is also a biased sample, which out of inspection, the repeatedly usage about the biased sample would have significant effect on reliability estimation. Therefore, we draw an additional sample when there are rejected samples of MCMC until there are an acceptable sample exists.

Figure 2.6 illustrates the procedure about the sample data filter via MCMC of accepting an additional sample. We summarize the procedure in algorithmic form as following:

1. Estimate statistical parameters of the underlying distribution by directly using existing samples as target distribution $\pi$.

2. Resample and estimate the statistical parameters by bootstrap method as the proposal distribution $q$.

3. Draw a sample $s^*$ from population, calculate the acceptance probability $a_k$ via the current sample $s_c$.

$$a_k(s_c, s^*) = \min\left(\frac{\pi(s^*)q(s^*, s_c)}{\pi(s_c)q(s_c, s^*)}, 1\right)$$

4. Generate $u$ from $U(0,1)$.

5. If $u \le a_k(s_c, s^*)$, then accept sample $s^*$ as additional sample.

6. Else, turn to step 3.

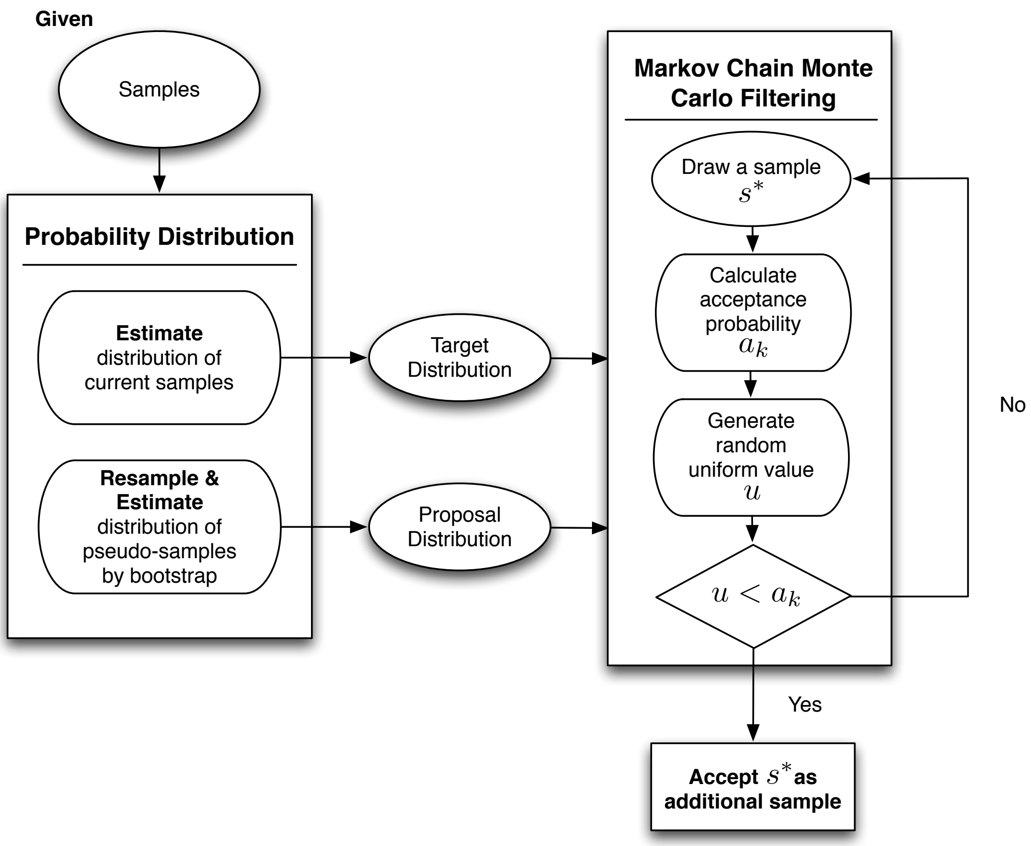



Figure 2.6: The procedure of MCMC of accepting an additional sample

# Chapter 3 Optimal Sampling Augmentation and Resource Allocation

In practical engineering applications, the underlying distribution about a specific uncertainty is not known. Limited samples from measurements or experimental results are generally the only available information of the uncertainty. In Chapter 2, we have described the fundamental concepts of using these samples to infer the distribution. In this section, we emphasis on how to cast additional samples to aid engineering decision-making in a reliability-based design optimization with limited samples. In this chapter, we introduce the optimization model of RBDO with inadequate uncertainty data, optimal sampling augmentation for design, and then the resource allocation.

## 3.1 RBDO with Inadequate Uncertainty Data

In this section, we focus on assist engineers to make decision in a reliability-based design optimization with limited samples. First, we will introduce reliability-based design optimization (RBDO), discuss about the activity of reliability constraint with inadequate uncertainty data, and construct the generalize optimization model of RBDO with inadequate uncertainty data.

### 3.1.1 Introduction of RBDO

In engineering design, the traditional deterministic optimization model has been successfully applied to systematically reduce the cost and improve quality. However, the existence of uncertainties in physical quantities such as manufacturing tolerances, material properties, and loads requires a reliability-based approach to design optimization [54,55]. Optimal design problems which consider uncertainties as random variables or parameters are formulated as problems with probabilistic constraints. Eq.(3.1) is a generalized single-objective probabilistic formulation with random design variables $\mathbf{D}$, random parameters $\mathbf{P}$, deterministic design variables $\mathbf{d}$ and deterministic parameters $\mathbf{p}$. The objective $f$ is a function of deterministic quantities and the mean values of all random quantities in the formulation and $\mathcal{K}$ is the constraint set. The deterministic feasible space of $\mathbf{d}$ subject to $\mathbf{g}(\mathbf{d}) \leq 0$ is $\mathcal{F}$.

$$\begin{aligned} &\min_{\boldsymbol{\mu}_{\mathbf{D}},\mathbf{d}} f(\boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{P}}, \mathbf{d}, \mathbf{p}) \\ &\Pr[g_j(\mathbf{D}, \mathbf{P}, \mathbf{d}, \mathbf{p}) > 0] \leq P_{\mathrm{f},j} \; \forall j \in \mathcal{K} \end{aligned} \tag{3.1}$$

Constraints with random variables and/or parameters are reformulated such that the probability of constraint $g_j$ violation is less than or equal to an acceptable failure limit $P_{\mathrm{f},j}$. Deterministic constraints (i.e., constraints that are not functions of any random quantities) are considered in the probabilistic form as a special case with the failure probabilities $P_{\mathrm{f},j}$ being 0. Equality constraints are not implicitly removed using the methods in [56]. This formulation is commonly referred to as reliability-based design optimization (RBDO) formulation in the literature [2].

$$\Pr[g(\mathbf{D}, \mathbf{P}) > 0] = \int \cdots \int_{g(\mathbf{D},\mathbf{P})>0} f_{\mathbf{DP}}(\mathbf{d}, \mathbf{p}) d\mathbf{d} d\mathbf{p} \tag{3.2}$$

Calculating probabilistic constraints in Eq.(3.1) requires a multiple integration over the failure domain as shown in Eq.(3.2) where $f_{\mathbf{DP}}$ is the joint probability density function (PDF) of all random uncertainties. However, the lack of joint PDF in most engineering problems and the difficulty in solving the multiple integration makes Eq.(3.2) impractical. Several methods have been proposed to improve the efficiency and accuracy of calculating constraint probabilities. Among them the first and the second order reliability methods (FORM and SORM) are most commonly used in engineering disciplines [3,57–59].

### 3.1.2 Activity of Bayesian Reliability Constraints

One of the underlying assumptions of Eq.(3.2) is the availability of the distribution function. In practice, a proper model of a random uncertain event requires a large amount of data and a proper selection of distribution type. The cost of generating these data could potentially be too high for the industry with limited resources and time to production. Therefore albeit with a large quantity of available literature on probabilistic optimization method, the industrial applications with actual products are rare.

In this work, instead of assuming distributions for uncertainties, we include uncertainty in the form of limited samples. Then we estimate the reliability via Bayesian inference as shown in Chapter 2. In this case, the RBDO problem becomes RBDO with inadequate uncertainty data.

The main different between the standard RBDO problems and the RBDO with inadequate uncertainty data is the evaluation of reliability constraint. Reliability constraint with inadequate uncertainty data is evaluated by Bayesian theory. In the section 2.2.4, we illustrated an example to show how to obtain the reliability distribution from samples. That example also demonstrated how to estimate reliability of a function with samples. Inadequate uncertainty data in the form of samples also exist when we develop a new product. We need to consider the reliability of our product on design stage, however, sometimes the underlying distributions of uncertain quantities are not available; instead, a few samples are the best we can hope for. The design problem with samples becomes a reliability-based design optimization with inadequate uncertainty data. Reliability constraints in RBDO need to be reformulated to account for samples. In the following, we will define Bayesian reliability constraint with samples and then study the activity of this constraint.

In the RBDO community, reliability constraint can be written as

$$\Pr[g \leq 0] \geq \text{reliability target} \tag{3.3}$$

Because of the inadequate uncertainty data of parameters, the probability $\Pr[g \leq 0]$, which referred as reliability, is not a fixed value, instead, it becomes a distribution $\mathbf{R}$ about the reliability value. Therefore, the constraint requires that reliability $\mathbf{R}$ being larger than a reliability target $R_t$ becomes a probability problem. Since it becomes a probability problem, we define the confidence range as Equation (2.27). Use the same concept as RBDO, the probability of reliability larger than the reliability target should reach a target confidence level as confidence range target $CR_t$. Then the final constraint formulation becomes

$$\begin{aligned} &\Pr\left[\Pr[g \leq 0] \geq R_t\right] \geq CR_t \\ &= \Pr\left[\mathbf{R} \geq R_t\right] \geq CR_t \end{aligned} \tag{3.4}$$

We define this constraint as Bayesian reliability constraint. In the remaining thesis, we will use Bayesian reliability constraint instead of Equation (3.3) as the reliability constraint for RBDO with inadequate uncertainty data.

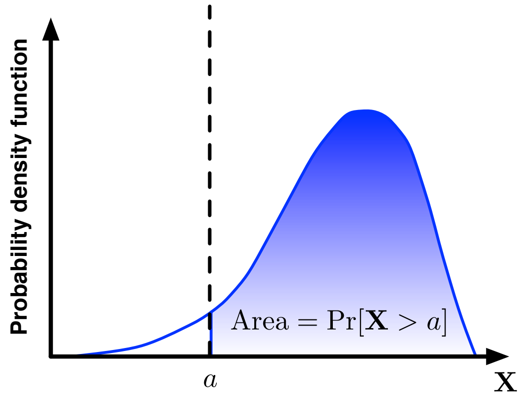



Figure 3.1: Equivalence constraint expression in percentile form

A Bayesian reliability constraint includes two probability concepts: (1) In standard optimization, a constraint is defined as active if removing the constraint changes the location of the optimum. (2) Mostly these active inequality constraints are strictly satisfied as equality when active. The activity of a Bayesian reliability constraint needs to be defined. Similar concepts should also apply to Bayesian reliability constraints. Let us look at a simple example of a random variable $X$. The constraint is defined as that the probability of $X$ being larger than a fixed value $a$ should be larger than $R_t$ as Figure 3.1

$$\Pr[X > a] \geq R_t \tag{3.5}$$

The equivalent expression is that $R_t \times 100$ percentile of $X$ is larger than $a$:

$$X^{(1-R_t)\times 100\%} \geq a \tag{3.6}$$

When the equivalence is set up, $X^{(1-R_t)\times 100\%} = a$, we say this constraint is “ active”.

Using the concept of equivalent probability constraint expression as in Equation (3.6), Equation (3.4) can also be expressed equivalently as

$$\mathbf{R}^{(1-CR_t)\times 100\%} \geq R_t \tag{3.7}$$

We define $\mathbf{R}^{(1-CR_t)\times 100\%}$ as Bayesian reliability $R_B$ in the remaining text to help us to use a fixed value to clarify the relationship between reliability constraint and confidence range and compare with reliability without inadequate uncertainty data.

We say that when $R_B = R_t$, the constraint is active. However, strictly satisfying as an equality is only the extends criterion of being an active constraint, we need also check whether the removal of a constraint will alter the location of the optimum. The importance of each constraint will be examined by these two definition of activities in this thesis.

### 3.1.3 Generalized Optimization Model of RBDO with Inadequate Uncertainty Data

Before constructing our optimization model, the form of design variables needs to be clarified. In this thesis, our focus is on uncertainties that could exist in the form distributions or in the form of available samples. In terms of design framework, uncertainties could be our design parameters or design variables. However, throughout this thesis, we consider uncertainties in design variables only be in the form of distributions. That said sample-type uncertainties are only in parameters.

Design variables are the quantities a designer pick to alter when updating a design. The values of these design variables change in a design process. Therefore samples taken based on previous uncertainty design variables can not be used to represent the the design variables that are about to change. In the literature, Gunawan et al. assumed a fixed Gaussian distribution with the mean being design variables, as shown in Figure 3.2 [17]; Picheny et al. used the bootstrap method to obtain a pseudo-distribution about their design variables [16]. Although they could include uncertainties of samples in design variables, their samples are converted into distributions of the variables instead of being used to calculate the reliability of the constraints. On the other hand, our uncertainties of samples in design parameters could better capture the

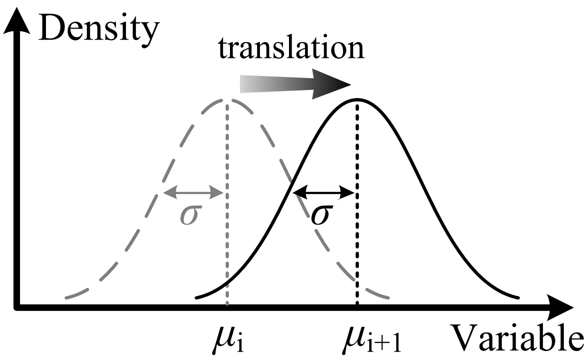



Figure 3.2: Schematic diagram about shift mean value of design variables

true engineering practice of parameters such as Young's modulus, road irregularity coefficient, oil density, and etc that we would not to design it but the characteristic of these parameters would affect the design decision.

In order to help designer use our optimal sampling augmentation process, we define the generalized model then the designer can utilize our sampling augmentation process easily. The generalized optimization model can be expressed as

$$\begin{aligned}
&\min_{\boldsymbol{\mu}_{\mathbf{D}_\mathrm{u}},\mathbf{d}} \ f(\boldsymbol{\mu}_{\mathbf{D}_\mathrm{u}}, \mathbf{d}, \mathbf{P}_\mathrm{s}, \boldsymbol{\mu}_{\mathbf{P}_\mathrm{u}}, \mathbf{p}) \\
&\text{s.t } g_i = g^i(\mathbf{d}, \mathbf{p}) \leq 0 \\
&\quad g_R = \Pr[g^R(\mathbf{D}_\mathrm{u}, \mathbf{d}, \mathbf{P}_\mathrm{u}, \mathbf{p}) \leq 0] \geq R_t \\
&\quad g_B = \Pr\left[\Pr[g^B(\mathbf{D}_\mathrm{u}, \mathbf{d}, \mathbf{P}_\mathrm{u}, \mathbf{P}_\mathrm{s}, \mathbf{p}) \leq 0] \geq R_t\right] \geq CR_t
\end{aligned} \tag{3.8}$$

where

$\mathbf{d}$ : deterministic design variables

$\mathbf{D}_\mathrm{u}$ : uncertain design variables known for distributions

$\mathbf{p}$ : deterministic parameters

$\mathbf{P}_\mathrm{u}$ : uncertain parameters known for distributions

$\mathbf{P}_\mathrm{s}$ : uncertain parameters known for samples

$g$ : deterministic constraint

$g_R$ : reliability constraints that constraint $g^R$ need to reach reliability target

$g_B$ : Bayesian reliability constraints that constraint $g^B$ need to satisfy the reliability target and confidence range target

Above model is the generalized reliability-based design optimization with inadequate uncertainty data. We need to classify the uncertainties types into parameters and design variables. The uncertain variables only exist in form of distributions as previous assertion. And the uncertain parameters could be classify into known for distributions and samples. The Bayesian reliability constraint $R_B$ is used to evaluate the reliability with inadequate uncertainty data. If Bayesian reliability constraint is removed, then the problem becomes RBDO problem.

## 3.2 Optimal Sampling Augmentation for Design

In this section, the proposed optimal sampling augmentation process will be illustrated. The purpose of sampling augmentation will also be explained.

### 3.2.1 Purpose of Sampling Augmentation

As discussed in Section 2.2.3 that concept of "sample combination" is made more clear and reasonable in practice than the concept of a set of all uncertainties. We cannot say that two sets of uncertain quantities with respect to two parameters from different measurement environment to be a set of samples because we do not know which one sample should be matched to another one.

As shown in Figure 2.2, with the increase of the number of samples, the confidence levels inferred will be higher. Therefore, in order to achieve a more creditable inference about an uncertainty, sample size need to reach a certain level. However, the resources to provide information about the uncertainties is limited. An effective reliability inference and optimization scheme should be provided under this limited situation. In existing research, the differentiations between the importance and cost associated with each uncertainty cannot be revealed due to the fact that samples exist in a group, rather than appear individually. As a result, we can not measure one uncertain parameter to increase number of samples when inferring the population. All uncertainties need to be measured. However, in our opinion, samples appear individually, resulting in " sample combination ". One measurement means one sample. Each samples is combined to form a set of sample combinations. The set of all possible uncertainty could also be obtained by the concept of sample combination. Therefore, we can compare the importance of each uncertainties.

The additional measurements can only be given on the critical uncertainties instead of measuring a set of whole uncertainties. We propose a sampling augmentation process that add samples based on their importance to reduce the cost of uncertainty measurements. Redundant measurements, the samples that do not add value in population inference, can be avoided. Resource can be allocated much more efficiently and effectively.

### 3.2.2 Sampling Augmentation Process

In our proposed approach, Bayesian binomial inference is used to obtain the reliability distribution with inadequate uncertainty data. Biased samples are filtered via MCMC with bootstrap. Figure 3.3 illustrates the flowchart of the proposed approach in dealing with uncertainties that are available as either samples or distributions. At the beginning, a reliability target for each constraint is given. An acceptable confidence range target is also given for Bayesian reliability constraints. The first major step in the flowchart is to identify two types of uncertainties,

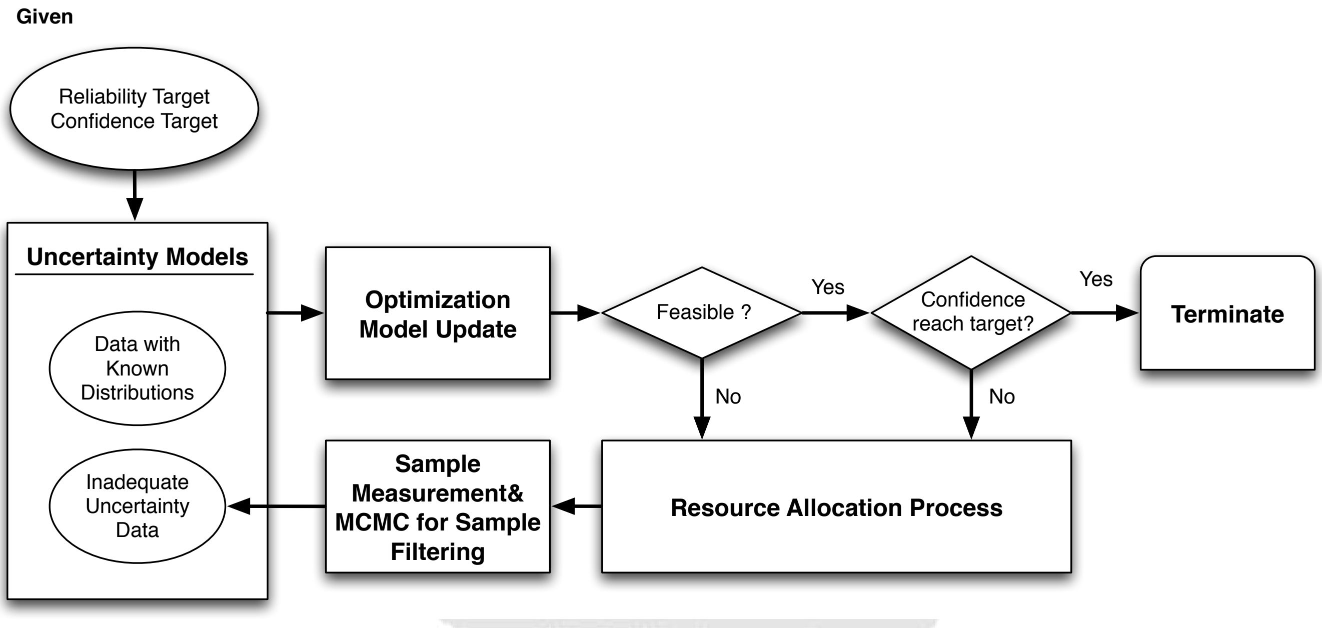



Figure 3.3: Flowchart of proposed sampling augmentation for design

namely the inadequate uncertainty data and the data with known distributions. We only consider uncertain design variables in the form of distributions. Uncertain parameters could be in distributions and in samples.

#### Optimization Model Update with Confidence Bound

Because the confidence bound will be updated with the increment of sample combinations, which is defined as confidence bound limit ($CBL$), the generalized optimization model demonstrate in Section 3.1.3 is changed with confidence bound limit as the following :

$$\begin{aligned}
&\min_{\boldsymbol{\mu}_{\mathbf{D}_{\mathrm{u}}},\mathbf{d}} \; f(\boldsymbol{\mu}_{\mathbf{D}_{\mathrm{u}}}, \mathbf{d}, \mathbf{P}_{\mathrm{s}}, \mathbf{P}_{\mathrm{u}}, \mathbf{p}) \\
&\text{s.t } g_i = g^i(\mathbf{d}, \mathbf{p}) \le 0 \\
&\quad g_R = \Pr[g^R(\mathbf{D}_{\mathrm{u}}, \mathbf{d}, \mathbf{P}_{\mathrm{u}}, \mathbf{p}) \le 0] \ge R_t \\
&\quad g_B = \Pr\left[\Pr[g^B(\mathbf{D}_{\mathrm{u}}, \mathbf{d}, \mathbf{P}_{\mathrm{u}}, \mathbf{P}_{\mathrm{s}}, \mathbf{p}) \le 0] \ge R_t\right] \ge CB
\end{aligned} \tag{3.9}$$

where the generalized optimization model alters with the change of confidence bound ($CB$). As shown in Section 2.2.2, the confidence range of a Bayesian reliability estimation is limited by the number of sample combinations. Therefore, after clarifying the uncertainties, we then evaluate the confidence bound of this number of sample combinations. This confidence bound is the maximal confidence range with the reliability target provided given existing number of sample combinations. The initial confidence bound is defined by the extreme state current samples could achieve. However, this extreme situation of confidence bound restricts the feasibility of constraint in optimization. If designers believe that the extreme situation of confidence bound would affect the feasibility of constraints, a certain degree of discount ($D_s$) on confidence bound is permitted. Then the discounted confidence bound ($CB$) is expressed as

$$CB = D_S(1 - \Phi_{\mathrm{Beta}}(R_t, N_c + 1, 1)) \tag{3.10}$$

where $D_s = 1$ means there is no relaxation on Bayesian reliability constraint. The relaxation means that we first allow the Bayesian reliability constraint could not achieve the confidence bound but give a quite closer design point. The relaxation level on Bayesian reliability constraint is decided by designer. With the sampling augmentation, the confidence bound limit would update with the increment of number of sample combinations. However, if previous sampling augmentation iteration obtained an infeasible design, the confidence bound limit would not be updated with the increment of number of sample combinations.

There is one thing needs to be noticed that the confidence bound limit becomes nearly 100% with large number of sample combinations. In other words, the optimum should satisfy a high confidence level (say 99.9%) to provide Bayesian reliability larger than the target reliability value with large sample combinations.

### MCMC for Sample Filtering

In practical engineering applications, limited samples are used to infer the population about an uncertainty. Although increasing sample size will improve the inference results, the accuracy of reliability estimation would also be affected by the quality of the measurements. Highly biased undesirable measurements would increase the number of design iterations required. A filter mechanism that effectively removes biased samples is necessary.

The proposed method filter mechanism includes : an MCMC of accepting the current sample and MCMC of accepting an additional sample. The MCMC can be divided into two parts : One part is the probability distributions, as shown in Figure 3.4, about the target distribution and proposal distribution. In order to give the filter judgment on the same standard, this two probability distributions are obtained from the initial samples. Another part is Markov chain Monte Carlo filtering. This part is used to examine the acceptance of additional samples. These two filter are shown in Figure 3.5. Figure 3.5(a) is about the filter mechanism of MCMC of accepting the current sample and Figure 3.5(b) is about the filter mechanism of MCMC of accepting an additional sample. These two filter mechanism will be used to filter out the biased samples. The detailed descriptions are as shown in Section 2.3.3.

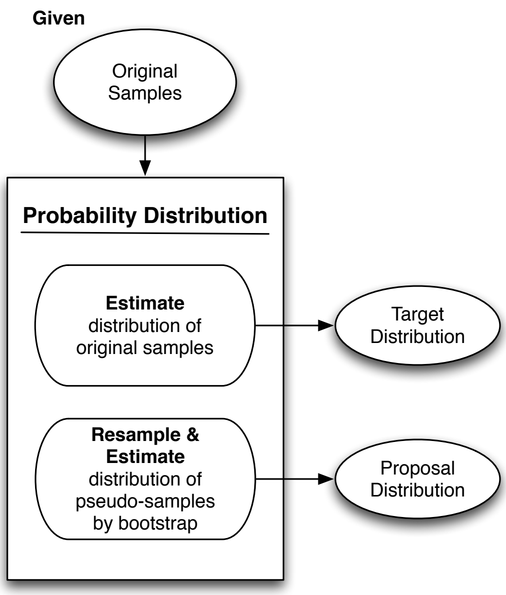
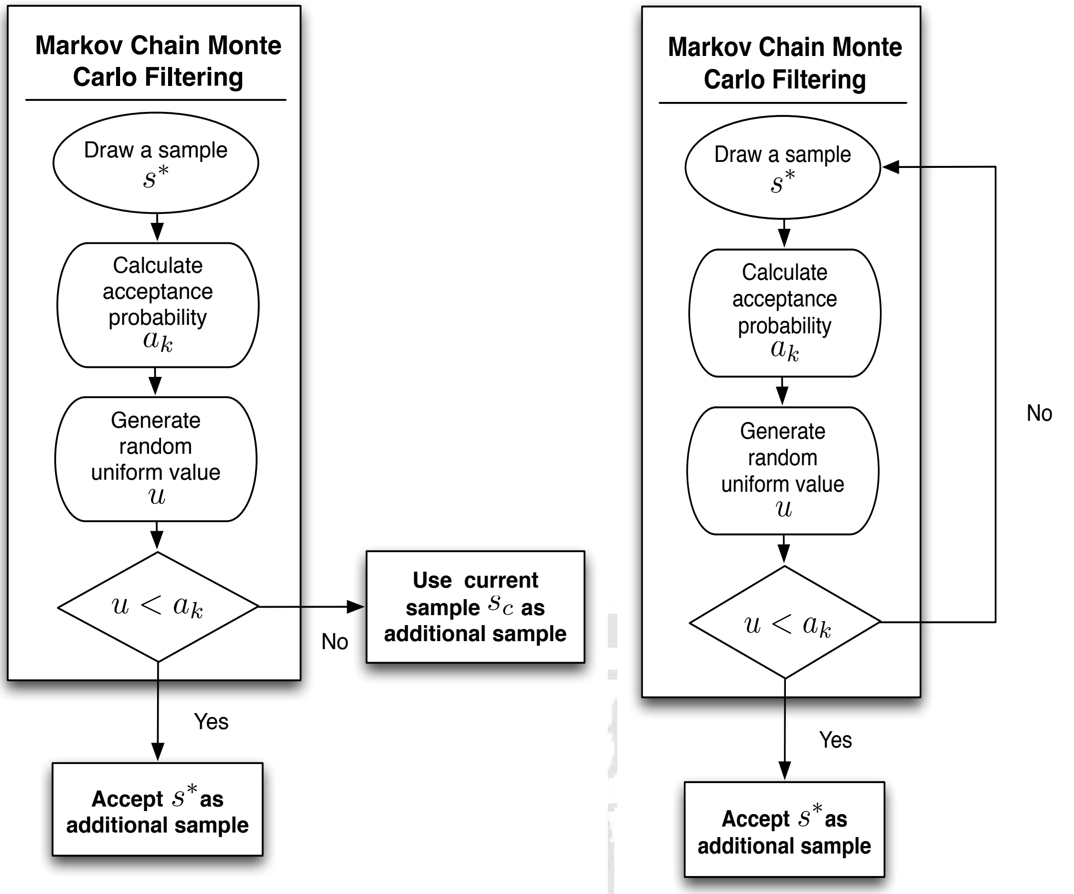



Figure 3.4: MCMC previous stage: probability distribution




(a) MCMC of accepting the current sample (b) MCMC of accepting an additional sample

Figure 3.5: Two MCMC filtering mechanism

### Algorithmic Steps of Sampling Augmentation for Design

Updating of the optimization model with confidence bound limit is the major steps in Figure 3.3. The flowchart of general sampling augmentation is as shown in Figure 3.3. The first step is to provide a reliability target, and an acceptable confidence range target for each constraint with uncertainties. Then, classify the uncertainties into the inadequate uncertainty data (samples) and data with known distributions. As shown in Section 2.2.3, we use the sample combinations to infer reliability distribution, therefore, we evaluate the number of sample combinations of these inadequate uncertainty data. Because the confidence range of a Bayesian reliability estimation is limited by the number of sample combinations, we use the confidence bound limit instead of confidence range target in the optimization model. Therefore, we evaluate and update the confidence bound limit of this number of sample combination in the optimization model. Then performing the reliability-based design optimization with inadequate uncertainty data to help us to find a better design under this number of sample combination. If there is feasible solution exists, then we check that whether the confidence range target is reached, the sampling augmentation will be terminated, if not, then we do resource allocation to find an important uncertainty. If there is no feasible solution exists, execute the resource allocation to give an important uncertainty. The detailed description about the resource allocation will be introduced in next section. Then we take an additional sample measurement then filter out the biased sample via MCMC sample data filter as Section 2.3.3 and 2.3.3 shown. The sampling augmentation will be terminated when the optimum's confidence range satisfying the confidence range target, otherwise, the process will continued to iterate. The general sampling augmentation can be expressed in algorithmic form as following :

**Step 1** Provide the reliability target value of each constraint with uncertainties. And the confidence range target of Bayesian reliability constraint is also given.

**Step 2** Classify the uncertainties into inadequate uncertainty data and data known for distributions. Evaluate the number of sample combinations of inadequate uncertainty data.

**Step 3** Evaluate and update the confidence bound limit of this number of sample combinations and update the optimization model. Perform reliability-based design optimization with inadequate uncertainty data.

**Step 4** Examine the existence of feasible solution.

- Feasible solutions exist : Examine the confidence range of this feasible design with confidence range target. If the confidence range target is reached, then the sampling augmentation process terminate, otherwise, go to step 5.
- No feasible solutions exist : Go to step 5.

**Step 5** Execute the resource allocation to give an important uncertainty. Go to step 6.

**Step 6** Give an additional sample measurement and filter the biased sample. Go to step 7.

**Step 7** The sampling augmentation process for design terminates when the optimal solution satisfying the confidence range target, otherwise, the process would continued iterates.

## 3.3 Resource Allocation Process

Measurements of samples can be costly. Unnecessary redundant measurements can be avoided by deliberated casting samples only when they are important. In Section 3.2, sampling augmentation and separating of each uncertainty help reducing the number of measurements required. Once additional sample are inevitable, in this work, measurements of uncertainties are added only when they are " important". Additional measurements on critical constraints with respect to uncertainties in the form of discrete sample are necessary. In what follows, two techniques of critically adding samples will be introduced. The proposed resource allocation scheme would then be illustrated.

### 3.3.1 Sensitivity Analysis

The constraint with the lowest confidence range in the generalized optimization model is the critical constraint. With this critical constraint, the sensitivity analysis is used to decide which uncertain parameter is the most important one with highest sensitivity.

Sensitivity analysis assesses the impacts in the change of a certain parameter on the overall system. Additional measurements would be given on critical constraints with respect to the most important uncertain parameters. However, critical constraints usually have more than one uncertain parameter. Therefore, we use sensitivity analysis to help us making decisions on which parameters are the key drivers of the critical constraints.

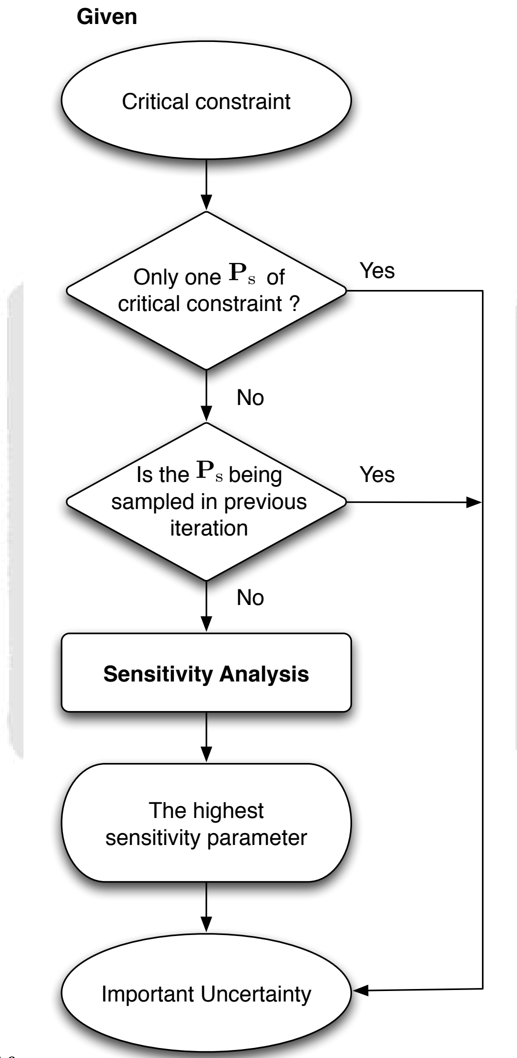
Let a constraint be $g$, with respect to two parameters $y$, and $z$. The sensitivity of $g$ with respect to each parameter value is the derivative with respect to each parameter. Define the sensitivity at the value of $(\bar{y}, \bar{z})$ as

$$\begin{aligned} s_y &= \left| \frac{\partial g}{\partial y} \right|_{y=\bar{y}} \\ s_z &= \left| \frac{\partial g}{\partial z} \right|_{z=\bar{z}} \end{aligned} \tag{3.11}$$

where $s_y$ is defined as sensitivity of parameter $y$, same as $s_z$. In this thesis, we evaluate the sensitivity of an uncertainty at the mean as its sensitivity to the function $g$.

### 3.3.2 Scheme of Resource Allocation

With proposed methods of sensitivity and MCMC for sample filtering, the overall resource allocation scheme is as shown in Figure 3.6. The sensitivity analysis is used to decide which uncertain parameter is more important when the critical constraint with respect to more than one uncertain parameters.




scheme.pdf

Figure 3.6: Flowchart of resource allocation scheme

First, we capture the constraint with lowest confidence range in generalized optimization model as the critical constraint. And we check if there is only one uncertain parameter known for samples ($\mathbf{P}_s$) with respect to the critical constraint. If there are only one uncertain parameter with respect to the critical constraint, we recognize this parameter is the only influential uncertain quantities, then regard this uncertainty as important uncertainty. Otherwise, if there are more than one uncertain parameters known for samples with respect to the critical constraint, we first check if the same uncertain parameter be draw in previous sampling augmentation iteration. If the same critical uncertain parameter exists, we make the assertion that this uncertain parameter is quite influential so we must give more measurement to help us make inference. On the contrary, if there is no same drawn uncertain parameter, the sensitivity analysis is used to decide which uncertain parameter is the most importance one with the highest sensitivity. When the important uncertain parameter with respect to the critical constraint is decided, the resource allocation process terminates.

The algorithmic form of resource allocation is provided as following:

**Step 1** Identify the constraint with the lowest confidence range as critical constraint.

**Step 2** Examine if there is only one uncertain parameter known for samples with respect to the critical constraint, if yes, then turn to step 5, otherwise to step 3.

**Step 3** Check if the same uncertain parameter has with respect to the critical constraint, then turn to step 5, else turn to step 4.

**Step 4** Perform sensitivity analysis to find which uncertain parameters with the highest sensitivity, then go to step 5.

**Step 5** Set the obtained uncertain parameter as important uncertainty.

# Chapter 4 Case Studies in Single Level Systems

Two case studies are used to show the effectiveness of the proposed approach in Chapter 3, which uses inadequate samples to assist reliability-based design. The optimization is solved by fmincon solver (sequential quadratic programming) in commercial tool Matlab. In this section, we focus on problems that is formulated as an all-in-one system. In Chapter 5, we will extend the concept to hierarchical complex problems. An all-in-one system is one where all objectives and constraints are handled in a single problem. Following, a mathematical example and passive vehicle suspension design are used to demonstrate the proposed approach in Section 4.1 and 4.2.

## 4.1 A Mathematical Example

A mathematical problem is used in this section to show the overall approach proposed. In comparison, we also study the same problem with different uncertainty levels, namely, deterministic problem with no uncertainty, RBDO problem with uncertainties known for distributions and RBDO problem with inadequate uncertainty data (uncertainties known for samples). The optimal results of these three design problems will be compared in following section. The optimal sampling augmentation for RBDO with inadequate uncertainty data will be demonstrated with three situations of different MCM filter mechanisms as without MCMC, MCMC of accepting current sample and MCMC of accepting an additional sample. These three types of sampling augmentation are compared with different sample size in Section 4.1.3

### 4.1.1 Optimization Model of Mathematical Example

#### Deterministic Optimization Model

$$
\begin{aligned}
&\min_{x} f = x \\
\text{s.t. } & g_1 = 1 - x p_1^2/20 \le 0 \\
& g_2 = 1 - (x + p_1 - 5)^2/30 \le 0 \\
& g_3 = 1 - 80/(p_1^2 + 8p_2 - 6.5) \le 0 \\
& g_4 = 1 - (x p_2 + p_2^2)/20 \le 0 \\
& g_5 = 1 - (x + p_1 + p_2 - 6)^2/30 - (-x + p_1 - p_2 - 11)^2/120 \le 0 \\
& 6.5 \le x \le 8
\end{aligned}
\tag{4.1}
$$

where parameters $p_1 = -8.2$ and $p_2 = 2.2$.

#### RBDO Model

$$
\begin{aligned}
&\min_{x} f = x \\
\text{s.t. } & g_1 = \Pr[1 - x\mathbf{P_1}^2/20 \le 0] \ge R_t \\
& g_2 = \Pr[1 - (x + \mathbf{P_1} - 5)^2/30 \le 0] \ge R_t \\
& g_3 = \Pr[1 - 80/(\mathbf{P_1}^2 + 8\mathbf{P_2} - 6.5) \le 0] \ge R_t \\
& g_4 = \Pr[1 - (x\mathbf{P_2} + \mathbf{P_2}^2)/20 \le 0] \ge R_t \\
& g_5 = \Pr[1 - (x + \mathbf{P_1} + \mathbf{P_2} - 6)^2/30 - (-x + \mathbf{P_1} - \mathbf{P_2} - 11)^2/120 \le 0] \ge R_t \\
& 6.5 \le x \le 8 \\
\text{where } & \begin{cases} \mathbf{P_1} \sim N(-8.2, 0.08^2) \\ \mathbf{P_2} \sim N(2.2, 0.02^2) \end{cases}
\end{aligned}
\tag{4.2}
$$

The reliability target is given as $R_t = 0.85$.

**RBDO Model with Inadequate Uncertainty Data**

$$
\begin{aligned}
&\min_{x} f = x \\
\text{s.t. } & g_1 = \Pr[\Pr[1 - x\mathbf{P_1}^2/20 \le 0] \ge R_t] \ge CB \\
& g_2 = \Pr[\Pr[1 - (x + \mathbf{P_1} - 5)^2/30 \le 0] \ge R_t] \ge CB \\
& g_3 = \Pr[\Pr[1 - 80/(\mathbf{P_1}^2 + 8\mathbf{P_2} - 6.5) \le 0] \ge R_t] \ge CB \\
& g_4 = \Pr[\Pr[1 - (x\mathbf{P_2} + \mathbf{P_2}^2)/20 \le 0] \ge R_t] \ge CB \\
& g_5 = \Pr[\Pr[1 - (x + \mathbf{P_1} + \mathbf{P_2} - 6)^2/30 - (-x + \mathbf{P_1} - \mathbf{P_2} - 11)^2/120 \le 0] \ge R_t] \ge CB \\
& 6.5 \le x \le 8
\end{aligned} \tag{4.3}
$$

The confidence bound ($CB$) will be updated with the increment of samples. The reliability target is given as $R_t = 0.85$. The confidence range target is given as $CR_t = 0.9$. The constraints' confidence bound limit of optimum must satisfy the confidence range target. The initial number of samples of each inadequate uncertainty data is five. The initial samples are given in Table 4.1. Table 4.2 shows the relationship between parameters in form of samples and constraints.

Table 4.1: 10 available initial data of $\mathbf{P}_1$ and $\mathbf{P}_2$ in Equation (4.3)

| $\mathbf{P_1}$ | $\mathbf{P_2}$ |
|---|---|
| -8.26284520912262 | 2.20222327815620 |
| -8.29582956598186 | 2.17863273276113 |
| -8.30545622577585 | 2.19175031781491 |
| -8.24754085360083 | 2.22269605270197 |
| -8.13573721029070 | 2.16495104270072 |

Table 4.2: Parameters respect to the constraints of the mathematical example

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| $\mathbf{P_1}$ | ✓ | ✓ | ✓ | | ✓ |
| $\mathbf{P_2}$ | | | ✓ | ✓ | ✓ |

### 4.1.2 Optimal Results and Discussions

The results of deterministic design, RBDO, and RBDO with inadequate uncertainty data would be compared. We will use Monte Carlo Simulation to acquire the reliability value of the optimal points, which referred as MCS reliability ( denoted as $R_{MCS}$) to represent as true reliability. About the RBDO with inadequate uncertainty data, the Bayesian reliability $R_B$ defined in Section 3.1.2 is used to represent the estimation value of the reliability distribution.

With intuitive, the lack of information would make the optimal results become conservative. Therefore, RBDO with inadequate uncertainty data should be the most conservative one, then RBDO be the second one and deterministic design be the last one. Table 4.3 shows the comparison of deterministic, RBDO, and RBDO with inadequate uncertainty data. The deduction could be proved in Table 4.3. The RBDO with inadequate uncertainty data is indeed the most conservative one about the function value $f$ in accordance of intuitive. In proposed sampling augmentation process, the resource allocation is considered by the critical constraint. As Table 4.3 show, the constraints $g_3$ and $g_4$ might be the critical constraints due to the lower reliability on RBDO optimum. As the result of MCMC of accepting current sample, the constraint $g_3$ is indeed the critical constraint. And the optimum of sampling augmentation with MCMC is quite close to the optimum of RBDO. The estimation of reliability by samples is denoted as Bayesian reliability $R_B$. Table 4.3 shows that the Bayesian reliability $R_B$ is quite close to the MCS reliability $R_{MCS}$. Therefore, we can say that the sampling augmentation process help us to use critically limited samples to obtain the credible reliability-based design.

Table 4.3: Comparison the results between RBDO, deterministic, and MCMC of mathematical example

<table>
<tr><th></th><th>Sampling Augmentation (accepting current sample)</th><th>RBDO</th><th>Deterministic</th></tr>
<tr><td>$x_{\mathrm{opt}}$</td><td>6.565137</td><td>6.5</td><td>6.5</td></tr>
<tr><td>$f$</td><td>6.565137</td><td>6.5</td><td>6.5</td></tr>
<tr><td>$R_B$</td><td>$(0.960, 0.960, 0.862, 0.960, 0.960)$</td><td rowspan="2">(1,1,0.894,0.869,1)</td><td rowspan="2">(1,1,0.894,0.869,1)</td></tr>
<tr><td>$R_{MCS}$</td><td>(1,1,0.894,0.961,1)</td></tr>
<tr><td>Active constraint</td><td>$g_3$</td><td>None</td><td>None</td></tr>
</table>

In Section 2.3.3, two types of MCMC filter mechanisms are proposed. The difference between these two filters is that when a new sample is rejected, one replicate the previous sample as a new one which the other one take a completely new measurement sample. Therefore, we have three scenarios in our study. Scenario 1 is sampling augmentation with MCMC of accepting current sample which means that when rejected sample occurs, the current sample be accepted as an additional sample. Scenario 2 is sampling augmentation with MCMC of accepting an additional sample which means that when rejected sample occurs, the filter mechanism will continue still an acceptable samples appear. Scenario 3 is sampling augmentation without MCMC filter mechanism. In the following, these three scenarios would be compared to show the effectiveness of MCMC filter. The comparison of these three scenarios is shown in Table 4.4.

Table 4.4: Comparison optimal results of three types of sampling augmentation of mathematical example

| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $x_{\text{opt}}$ | 6.565137 | 6.565130 | 6.565139 |
| $f$ | 6.565137 | 6.565130 | 6.565139 |
| $R_B$ | (0.96,0.96,0.86,0.96,0.96) | (0.96,0.96,0.86,0.96,0.96) | (0.96,0.96,0.85,0.96,0.96) |
| $R_{MCS}$ | (1,1,0.894,0.961,1) | (1,1,0.894,0.961,1) | (1,1,0.894,0.961,1) |
| Confidence range | (1,1,0.937,1,1) | (1,1,0.937,1,1) | (1,1,0.911,1,1) |
| Adding procedure | 6 on $\mathbf{P_1}$ | 6 on $\mathbf{P_1}$(actually 8 samples) | 7 on $\mathbf{P_1}$ |
| No.rejected sample | 2 | 2 | N/A |
| No. combination | 55 | 55 | 60 |
| Active constraint | $g_3$ | $g_3$ | $g_3$ |

Different sampling augmentation scheme obtain similar optimal results. Both the reliability estimation ($R_B$) and confidence range($CR$) reach the target. The estimation of reliability ($R_B$) of three scenarios are quite close to $R_{MCS}$ reliability of overall constraints. Under this situation, we can say that Bayesian binomial inference provide a credible estimation about the reliability distribution. Both sampling augmentation with MCMC filter mechanism reject two biased samples and use 6 additional samples on $\mathbf{P_1}$ in optimization model. The overall sample size of scenarios 2 is 8 (6+2) samples. As shown in Table 4.4, number of sample combinations of both two sampling augmentation with MCMC (scenario 1 and 2) is less than of which without MCMC (scenario 3). Sampling augmentation without MCMC obtain the optimum requires more sampling iterations.

Confidence range is the likelihood probability of the estimation of reliability distribution being greater than a reliability target which is limited by the number of sample combinations. Figure 4.1 shows the confidence range of constraints $g_3$. With increment of number of sample combinations, the confidence should be increased. However, biased samples could undermine the confidence range value without MCMC. The 4th iteration in Figure4.1 comparing with and without MCMC shows a big difference in confidence range calculation. The biased sample would also baffle the search of the optimal point. Overall, biased samples affect the convergency of the RBDO with inadequate uncertainty data. From the results, the MCMC filter mechanism could assist the convergent rate of sampling augmentation for design as Figure 4.1 shown and use fewer samples to inference the reliability distribution.

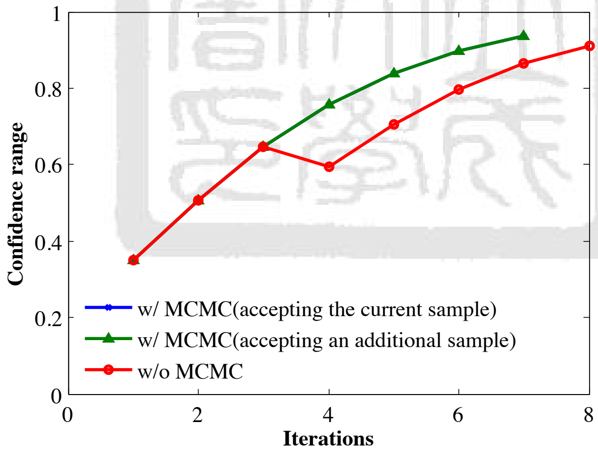
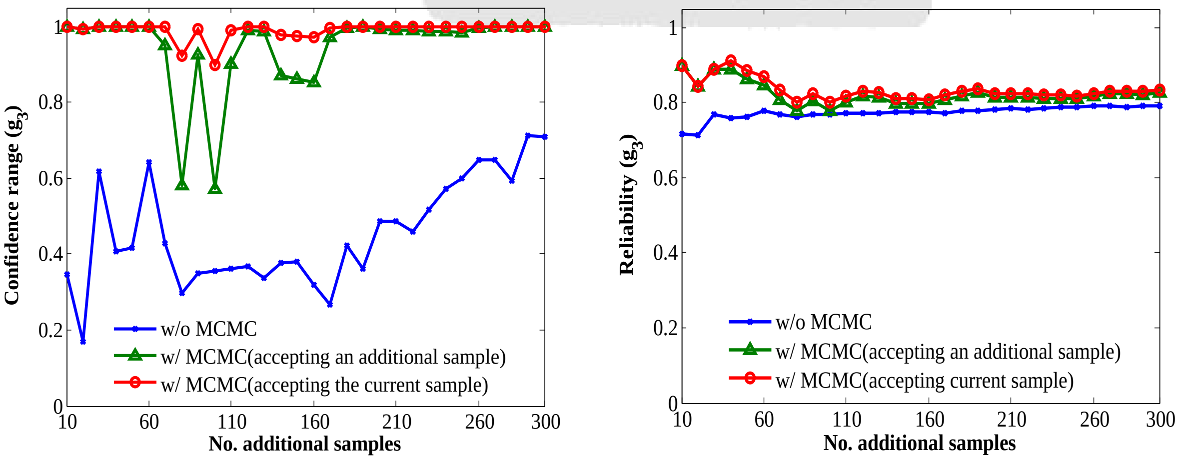



Figure 4.1: Confidence range of $g_3$ of iterations

### 4.1.3 Comparison of Sampling Augmentation with Different Sample Size

The effect of MCMC filter mechanism on the same number of additional samples will be demonstrated in this section. Bayesian reliability values and corresponding confidence ranges on the critical constraint $g_3$ with different sample size in the mathematical example in Equation (4.3) will be studied. The number of rejected samples is also taken into account as the number of additional samples. Figure 4.2 shows the comparison of MCMC of accepting current sample, MCMC of accepting an additional sample and without MCMC on optimal point of RBDO.

As shown in Figure 4.2(a), we can see the confidence range of $g_3$ without MCMC fluctuates up and down due to the effects of biased samples. Biased sample makes the confidence range of $g_3$ without MCMC unstable. Search directions in optimization also become inconsistent due to the fluctuations of confidence ranges. Then the convergency of optimization becomes slow. From the results, we can assert that the variation of confidence range value would make the search of optimum become difficult. The Bayesian reliability of $g_3$ without MCMC would no longer larger than reliability target ($R_t = 0.85$) due to the biased samples as shown in Figure 4.2(b). Therefore, we assert that both MCMC filter mechanisms would assist the effect to avoid biased samples and improve the convergency of optimization.




(a) Confidence range of $g_3$

(b) Bayesian reliability of $g_3$

Figure 4.2: Comparison three sampling augmentation process with different sample size on RBDO optimal point

## 4.2 Passive Vehicle Suspension Design

In this section, a passive vehicle suspension design is used to show the overall approach proposed. we will construct the optimization models about passive vehicle suspension design. In comparison, we also study the same problem with different uncertainty levels, namely, deterministic problem with no uncertainty, RBDO problem with uncertainties known for distributions and RBDO problem with inadequate uncertainty data (uncertainties known for samples). The optimal results of these three design problems will be compared in following section. The optimal sampling augmentation for RBDO with inadequate uncertainty data will be demonstrated with three situations of different MCM filter mechanisms as without MCMC, MCMC of accepting current sample and MCMC of accepting an additional sample. These three types of sampling augmentation are compared with different sample size in Section 4.2.3.

### 4.2.1 Optimization Model of Passive Vehicle Suspension Design

The optimal design of a passive vehicle suspension, shown in Figure 4.3, is studied following Lu *et al.* [60]. The objective is to minimize the mean square value of the vertical vibration acceleration of the vehicle body, which satisfies the following constraints: a lower bound on the road-holding ability of the vehicle ($g_1$); an upper bound on the rolling angle ($g_2$); a lower bound on the suspension's dynamic displacement to avoid bumper hitting, the so-called rattle-space constraint ($g_3$); and a lower bound on tire stiffness because tire life is an increasing function of tire stiffness ($g_4$).

Suspension stiffness $c$ (kg/cm), tire stiffness $c_k$ (kg/cm) , and damping coefficient $k$ (kg/cm/sec) are the design variables. The problem parameters are provided in Table 4.5.

In comparison, we study the same problem with different uncertainty levels, namely, deterministic problem with no uncertainty, RBDO problem with uncertainties known for distributions and RBDO problem with inadequate uncertainty data (uncertainties known for samples). Following, we construct the optimization models for these three design problems.

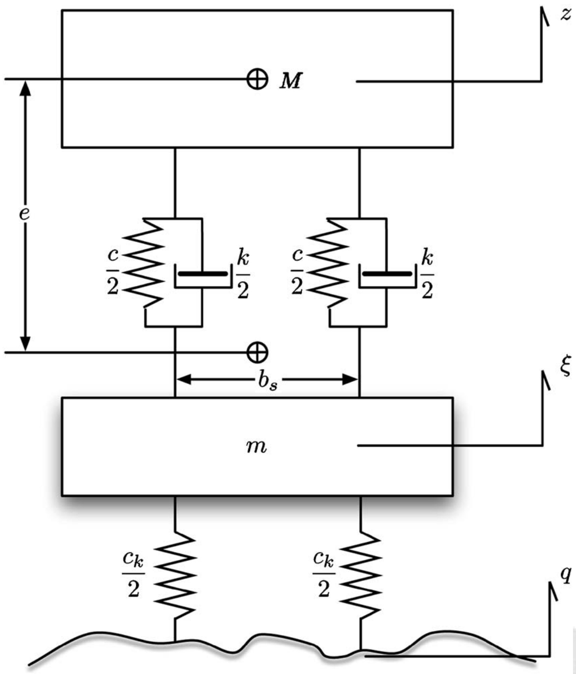



Figure 4.3: Passive vehicle suspension

**Deterministic Optimization Model**

$$\begin{aligned}
&\min_{c,c_k,k} \ddot{Z}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2k^{-1}) \\
&\text{s.t.} \\
&g_1 = \left(\frac{\pi AVm}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{M+m} - \frac{c}{M}\right)^2 + \frac{c^2}{Mm} + \frac{c_k k^2}{mM^2}\right) - 1 \leq 0 \\
&g_2 = 7.6394(4000(Mg)^{-1.5}c - 1) - 1 \leq 0 \\
&g_3 = 0.5(Mg)^{1/2}(k^2 c_k c^{-1}(M+m)^{-1} + c)^{-1/2} - 1 \leq 0 \\
&g_4 = ((M+m)g)^{0.877}c_k^{-1} - 1 \leq 0
\end{aligned} \tag{4.4}$$

Table 4.5: Suspension problem parameters

| | |
|---|---|
| Dynamic load coefficient, $b_0$ | 0.27 |
| Vehicle velocity, $V$ (m/s) | 10 |
| Gravity acceleration, $g$ (cm/s$^2$) | 981 |
| Road irregularity coefficient, $A$ (cm$^2$ cycle/m) | 1 |
| Sprung mass, $M$ (kg/cm/s$^2$) | 3.2 |
| Unsprung mass, $m$ (kg/cm/s$^2$) | 0.8 |

**RBDO Model**

$$
\begin{aligned}
&\min_{c,c_k,k} \ddot{Z}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2 k^{-1}) \\
&\text{s.t.} \\
g_1 &= \Pr\left[\left(\frac{\pi \mathbf{A} V \mathbf{m}}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{\mathbf{M}+\mathbf{m}} - \frac{c}{\mathbf{M}}\right)^2 + \frac{c^2}{\mathbf{M}\mathbf{m}} + \frac{c_k k^2}{\mathbf{m}\mathbf{M}^2}\right) - 1 \le 0\right] \ge R_t \\
g_2 &= \Pr[7.6394(4000(\mathbf{M}g)^{-1.5}c - 1) - 1 \le 0] \ge R_t \\
g_3 &= \Pr[0.5(\mathbf{M}g)^{1/2}(k^2 c_k c^{-1}(\mathbf{M}+\mathbf{m})^{-1} + c)^{-1/2} - 1 \le 0] \ge R_t \\
g_4 &= \Pr[((\mathbf{M}+\mathbf{m})g)^{0.877} c_k^{-1} - 1 \le 0] \ge R_t
\end{aligned}
\tag{4.5}
$$

The reliability target is given as $R_t = 0.9$. The road irregularity $\mathbf{A}$, sprung mass $\mathbf{M}$ and unsprung mass $\mathbf{m}$ are set to be uncertain parameters. The distributions are followed Normal distribution as

$$
\text{where} \begin{cases} \mathbf{A} \sim N(1, 0.03^2) \\ \mathbf{M} \sim N(3.2, 0.03^2) \\ \mathbf{m} \sim N(0.8, 0.005^2) \end{cases}
$$

**RBDO Model with Inadequate Uncertainty Data**

$$\min_{c,c_k,k} \ddot{Z}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2 k^{-1})$$

s.t.

$$g_1 = \Pr\left[\Pr\left[\left(\frac{\pi \mathbf{A} V \mathbf{m}}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{\mathbf{M}+\mathbf{m}} - \frac{c}{\mathbf{M}}\right)^2 + \frac{c^2}{\mathbf{Mm}} + \frac{c_k k^2}{\mathbf{mM}^2}\right) - 1 \le 0\right] \ge R_t\right] \ge CB \quad (4.6)$$

$$g_2 = \Pr[\Pr[7.6394(4000(\mathbf{M}g)^{-1.5}c - 1) - 1 \le 0] \ge R_t] \ge CB$$

$$g_3 = \Pr[\Pr[0.5(\mathbf{M}g)^{1/2}(k^2 c_k c^{-1}(\mathbf{M}+\mathbf{m})^{-1} + c)^{-1/2} - 1 \le 0] \ge R_t] \ge CB$$

$$g_4 = \Pr[\Pr[((\mathbf{M}+\mathbf{m})g)^{0.877} c_k^{-1} - 1 \le 0] \ge R_t] \ge CB$$

The road irregularity **A**, sprung mass **M** and unsprung mass **m** are set to be uncertain parameters of samples. The confidence bound ($CB$) will be updated with the increment of samples. The reliability target is given as $R_t = 0.9$. The confidence range target is given as $CR_t = 0.9$. The constraints' confidence bound limit of optimum must satisfy the confidence range target.The initial number of samples of each inadequate uncertainty data is five. The origin samples are given in Table 4.6. Table 4.7 shows the relationship between parameters in form of samples and constraints.

Table 4.6: 15 samples as initial uncertainty data of passive vehicle suspension design in Equation (4.6)

| **A** | **M** | **m** |
|---|---|---|
| 1.02021096089625 | 3.18716771531861 | 0.806528115517652 |
| 0.979926610138818 | 3.18261880578225 | 0.804919847656519 |
| 0.987990318980960 | 3.22777905144734 | 0.793743068217506 |
| 0.979845927183428 | 3.20016531224747 | 0.799101231046973 |
| 1.01726887049749 | 3.18096527441576 | 0.796282967758515 |

### 4.2.2 Optimization Result and Discussions

The results of deterministic design, RBDO, and RBDO with inadequate uncertainty data would be compared. We will use Monte Carlo Simulation to acquire the reliability value of the optimal

Table 4.7: Parameters respect to the constraints of the passive vehicle suspension design

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| **A** | ✓ | | | |
| **M** | ✓ | ✓ | ✓ | ✓ |
| **m** | ✓ | | ✓ | ✓ |

points, which referred as MCS reliability ( denoted as $R_{MCS}$) to represent as true reliability. About the RBDO with inadequate uncertainty data, the Bayesian reliability $R_B$ defined in section 3.1.2 is used to represent the estimation value of the reliability distribution.

Table 4.8 shows the comparison of deterministic, RBDO, RBDO with inadequate uncertainty data. Deterministic design is assumed that there are no uncertainties, the optimum should be more affirmatory than which with uncertainties. RBDO with inadequate uncertainty data would be the most conservative of these three design optimization problem. Because the reliability in RBDO with inadequate uncertainty data is also a uncertain quantities, the reliability estimation should confirm certain degree of confidence level (which means confidence range in this thesis). In order to confirm the confidence range of reliability distribution, the optimum becomes conservative in RBDO with inadequate uncertainty data. As shown in Table 4.8, the RBDO with inadequate uncertainty data is the most conservative one about the mean square value of the vertical vibration acceleration $\ddot{Z}^2$. Although RBDO with inadequate uncertainty data is the most conservative one, it still gives a optimal value closer to which of RBDO. In practical engineering community, the characteristics of underlying distribution cannot be known, what we can do is that only draw samples from population and infer the underlying distribution. We use $10^6$ pseudo-samples to simulate the underlying distribution in RBDO problem. If we want to assume the underlying distribution well-known, we must draw $10^6$ samples from population. However, measuring samples is costly, the resources to provide information about the uncertainties is limited. Optimal sampling augmentation with MCMC of accepting current sample only uses 20 samples(number of initial samples = 15, number of additional samples =5) to provide an acceptable optimal results. Summarizing, sampling augmentation process help us to use a small amount of samples to give a creditable reliability-base design.

Table 4.8: Comparison the results between RBDO, deterministic, and MCMC of Passive Vehicle Suspension Design Optimization

| | Sampling Augmentation (accepting current sample) | RBDO | Deterministic |
|---|---|---|---|
| $[c, c_k, k]_{\text{opt}}$ | $[393.2, 1437, 20.97]$ | [386.5,1469,20.77] | [379.8,1426,20.77] |
| $\ddot{Z}^2$ | 2915154 | 2911614 | 2819585 |
| $R_B$ | $(0.991, 0.991, 0.963, 0.963)$ | (0.997,0.997,0.906,1) | (0.984,1,0.0625,1) |
| $R_{MCS}$ | (1,1,0.874,0.899) | | |
| Confidence range | (1,1,1,1) | N/A | N/A |
| Adding procedure | 5 on **M** | N/A | N/A |
| No.rejected sample | 2 | N/A | N/A |
| No. combination | 250 | N/A | N/A |
| Active constraint | $g_3, g_4$ | $g_3$ | $g_2, g_3, g_4$ |

In Section 2.3.3, two types of MCMC filter mechanisms are proposed. The difference between these two filters is that when a new sample is rejected, one replicate the previous sample as a new one which the other one take a completely new measurement sample. Therefore, we have three scenarios in our study. Scenario 1 is sampling augmentation with MCMC of accepting current sample which means that when rejected sample occurs, the current sample be accepted as an additional sample. Scenario 2 is sampling augmentation with MCMC of accepting an additional sample which means that when rejected sample occurs, the filter mechanism will continue still an acceptable samples appear. Scenario 3 is sampling augmentation without MCMC filter mechanism. In the following, these three scenarios would be compared to show the effectiveness of MCMC filter. The comparison of these three scenarios is shown in Table 4.9.

As Section 4.1.3 shwon, we assert that the MCMC filter would help the convergent rate of optimization. Sampling augmentation without MCMC cannot give a feasible solution by 50 additional samples as Table 4.9 shown. Because the biased samples would let the Bayesian reliability constraint become unstable, the confidence range of reliability distribution would fluctuate due to biased samples. MCMC filter provides a stable confidence range of reliability estimation by filtering biased samples, it makes the searching direction would not be affected

Table 4.9: Comparison optimal results of three types of sampling augmentation of Passive Vehicle Suspension Design

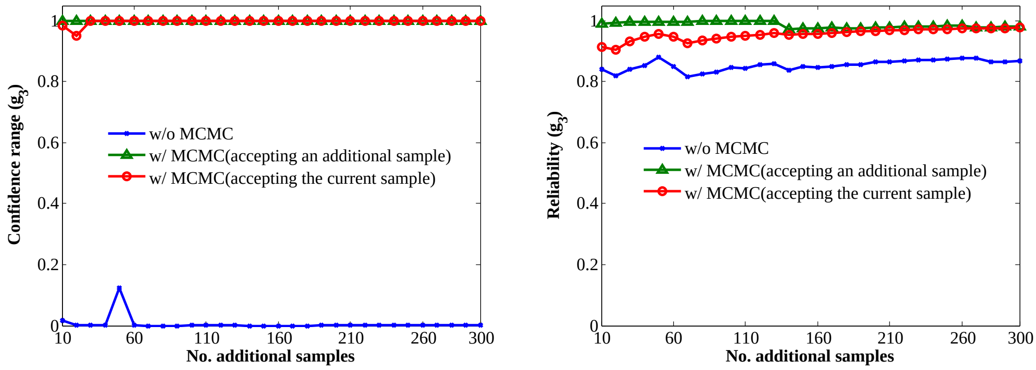
| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $[c, c_k, k]_{\text{opt}}$ | $[393.2, 1437, 20.97]$ | $[393.2, 1437, 20.97]$ | |
| $\ddot{Z}^2$ | 2915154 | 2915154 | |
| $R_B$ | $(0.991, 0.991, 0.963, 0.963)$ | (0.991,0.991,0.963,0.963) | |
| $R_{MCS}$ | (1,1,0.874,0.899) | (1,1,0.874,0.899) | |
| Confidence range | (1,1,1,1) | (1,1,1,1) | Diverge by 50 samples |
| Adding procedure | 5 on **M** | 5 on **M**(actually 9 samples) | |
| No.rejected sample | 2 | 4 | |
| No. combination | 250 | 250 | |
| Active constraint | $g_3, g_4$ | $g_3, g_4$ | |

by fluctuated confidence range values in optimization.

Scenario 1 and 2 give the same feasible solutions. Scenario 1 and 2 use five additional samples, but actually scenario 2 uses nine samples. Scenario 2 rejects two more samples than which of scenario 1. Actually, scenario 1 replicate the previous sample as a new one when a new sample is rejected. We can see that scenario 1 use fewer samples than which scenario 2 uses. Therefore, it is made the assertion that sampling augmentation with MCMC of accepting current sample (scenario 1) is the best one in three sampling augmentation processes on the aspect of amount of samples.

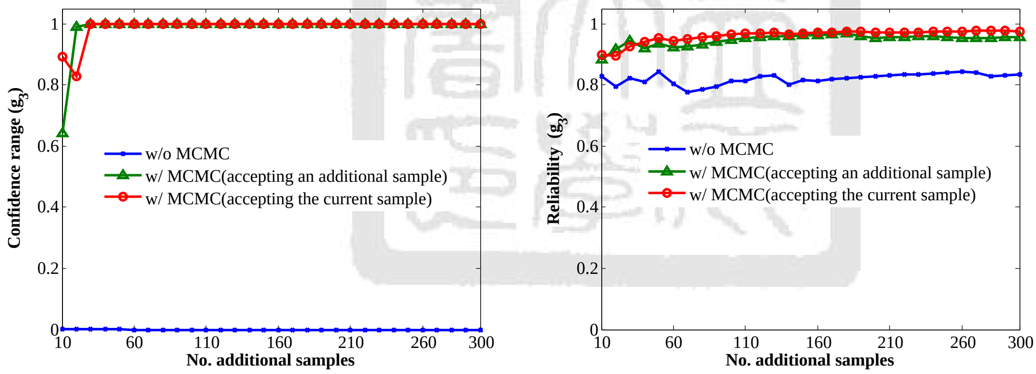
### 4.2.3 Comparison of MCMC and without MCMC in Passive Vehicle Suspension Design with different sample size

The effect of MCMC filter mechanism on the same number of additional samples will be demonstrated in this section. Bayesian reliability values and corresponding confidence ranges on the critical constraint $g_3$ with different sample size in the mathematical example in Equation (4.6) will be studied. The number of rejected samples is also taken into account as the number of additional samples. Figure 4.4 shows the comparison of MCMC of accepting current sample, MCMC of accepting an additional sample and without MCMC on optimal point of RBDO in Table 4.9. Figure 4.5 shows the comparison on optimum of sampling augmentation with MCMC filter in Table 4.9.




(a) Confidence range of $g_3$ (b) Reliability of $g_3$

Figure 4.4: Comparison three sampling augmentation process with different sample size on RBDO optimal point




(a) Confidence range of $g_3$ (b) Reliability of $g_3$

Figure 4.5: Comparison three sampling augmentation process with different sample size on MCMC(accept as current) optimal point

As shown in Figure 4.4(a) and 4.5(a), we can see the confidence range of $g_3$ without MCMC are almost near 0 that means there are no probability to get reliability higher than 90%. As shown in Figure 4.4(b) and 4.5(b), we can see the reliability of $g_3$ without MCMC fluctuates due to the effects of biased samples. The variation of constraint value would make the direction of searching optimum always changed in different iterations. Then the convergency of optimization becomes slow or divergent. As Figure 4.4 and 4.5 shown, both two MCMC filters have higher stability than which without MCMC. Therefore, we can make assertion that the MCMC filter mechanisms give the assist to sampling augmentation to avoid biased samples and improve the convergency of optimization.

## 4.3 Summary

We measure additional samples to help us to comprehend the importance of different uncertainties by sampling augmentation during optimal iterations. Therefore, we can only use lower fund to obtain the measurements of samples. Measurement is expensive, so resource allocation is necessary. We only measure additional samples when the corresponding uncertainty is important. However, measurements sometimes will go wrong, the filter mechanism MCMC is used to avoid higher biased samples. In this two case studies, the sampling augmentation with MCMC provides a creditable optimum which is quite closer to the optimum of RBDO by limited samples. Both two filter mechanisms give significant effect on filtering out biased samples, which assisting the convergency of optimization. The reliability estimation by Bayesian binomial inference is quite close to the MCS reliability, we can assert that the concept of sample combinations help us to reveal all possible situations that each sample be matched. We also examine the influence of biased samples by comparing the filter mechanism with different sample size. We see the phenomena that the biased samples would affect the consistency of constraint values. Use the filter mechanism MCMC let us avoid the inconsistency of constraint values then refrain the divergency of optimization.

# Chapter 5 Case Studies in Complex Multilevel System

Design problem is always solved by the all-in-one (AiO) strategy that consider the overall design all together. However, design of modern engineering product becomes a complex system design problem. Furthermore, in practical engineering community, no single group could handle a complex design problem. The designers are always distributed over different design groups that independently make the proper design decisions [61]. The reliability design with inadequate uncertainty data is also existing in complex systems. In order to confirm that the proposed sampling augmentation and resource allocation for design can also be applied to the complex system design, we would demonstrate passive vehicle design same as in Section 4.2 with the introduction of design details of spring and damper.

## 5.1 Introduction to Analytical Target Cascading

The complex system design can be solved using decomposition strategies. The original AiO problem is partitioned into several subproblems. The goal of these strategies is to obtain the same solution as which with AiO formulations. Many different strategies are proposed such as Optimization by Linear Decomposition (OLD) [62], Quasi-separable Subsystem Decomposition (QSD) [63], Bi-Level Integrated Synthesis (BLISS) [64] and Collaborative Optimization (CO) [65]. Such methods are collectively referred to as multidisciplinary design optimization (MOD) methods. Another famous method to solve the complex design problem is Analytical Target Cascading (ATC) [66]. In this work, we will focus on analytical target cascading method.

Analytical target cascading is a model-based, multi-level, hierarchical optimization method for system design. Design targets from higher level subproblems are cascaded down to the lower level subproblems. ATC provides the multi-level formulation. Several variants of ATC have been proposed, we will focus on one type of these variants. First, the decomposition procedure for ATC is presented. Then the selected variant of ATC, augmented Lagrangian method for ATC [67] is presented.

### 5.1.1 ATC Problem Decomposition

ATC strategy provides a hierarchic multi-level formulation of the complex optimization problem. ATC formalizes the process of propagating top-level targets throughout the design hierarchy. An all-in-one design problem can be decomposed into several element as Figure 5.1. The top-level element handle the overall system design and each lower level elements are presented a subsystem or a component of its parent element. The elements are coupled by response variables and targets from parent. The optimization model of a subsystem is formulated by the local variables, response variables to parent, and targets to children which can minimize the inconsistency of response variables. The response variables would be iteratively rebalanced up to higher level element to achieve the consistency.

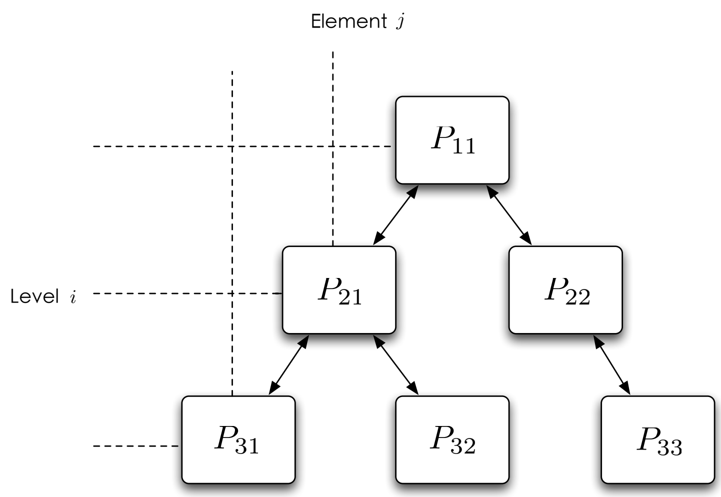



Figure 5.1: Scheme of hierarchic structure

The mathematical definition of the $j$-th subproblem at the $i$-level, namely, subsystem $P_{ij}$

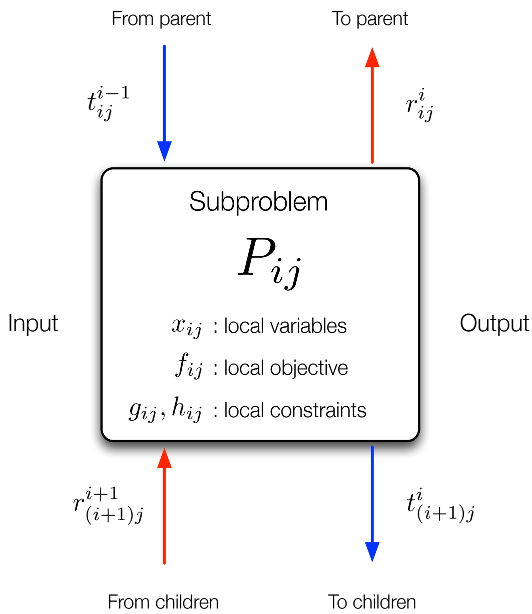



Figure 5.2: Subproblem flow in the ATC formulation

in Figure 5.2, is defined as follows.

$$
\begin{aligned}
&\min_{\bar{\mathbf{x}}}(f_{ij}(\bar{\mathbf{x}}_{ij}) + ||\mathbf{t}_{ij}^{i-1} - \mathbf{r}_{ij}^{i}|| + ||\mathbf{t}_{(i+1)j}^{i} - \mathbf{r}_{(i+1)j}^{i+1}||) \\
&\text{subject to} \\
&\qquad g_{ij}(\bar{\mathbf{x}}_{ij}) \leq 0 \\
&\qquad h_{ij}(\bar{\mathbf{x}}_{ij}) = 0 \\
&\text{where} \\
&\qquad \bar{\mathbf{x}}_{ij} = [\mathbf{x}_{ij}\ \mathbf{r}_{ij}^{i}\ \mathbf{t}_{(i+1)j}^{i}]
\end{aligned}
\tag{5.1}
$$

Here $\mathbf{t}_{ij}^{i-1}$ are the targets coming from the parent subproblem at level $i-1$, $\mathbf{r}_{ij}^{i}$ are the responses to be sent to the parent subproblem, $\mathbf{t}_{(i+1)j}^{i}$ are the targets to the children subproblems and $\mathbf{r}_{(i+1)j}^{i+1}$ are the responses from the children subproblems. The linking variables $\mathbf{t}_{ij}^{i-1}$ and $\mathbf{r}_{(i+1)j}^{i+1}$ from parent and children, respectively, are considered as parameters in the subproblem $P_{ij}$.

### 5.1.2 Augmented Lagrangian Method for ATC

In previous section, the mathematical model of ATC strategy is presented. Tosserams *et al.* [67] propose the augmented Lagrangian method to improve the convergent rate of the ATC strategy. This method use the augmented Lagrangian penalty function $\pi_{\mathrm{AL}}$ as Equation (5.2)

$$\pi_{\mathrm{AL}} = \mathbf{v}_{ij}(\mathbf{r}_{ij}^{i} - \mathbf{r}_{ij}^{i-1}) + ||\mathbf{w}_{ij} \circ (\mathbf{r}_{ij}^{i} - \mathbf{r}_{ij}^{i-1})||_2^2 \tag{5.2}$$

where the $\mathbf{v}_{ij}$ is a vector of Lagrangian multiplier parameters, the $\mathbf{w}_{ij}$ is a vector of penalty weights and the $\circ$ symbol is used to denote a term-by-term multiplication of vectors such that $[a_1, a_2, \ldots, a_n] \circ [b_1, b_2, \ldots, b_n] = [a_1b_1, a_2b_2, \ldots, a_nb_n]$.

Then Equation (5.1) becomes as following

$$\begin{aligned} & \min_{\bar{\mathbf{x}}} (f_{ij}(\bar{\mathbf{x}_{ij}}) + \pi_{\mathrm{AL}}) \\ \text{subject to} & \\ & g_{ij}(\bar{\mathbf{x}_{ij}}) \leq 0 \\ & h_{ij}(\bar{\mathbf{x}_{ij}}) = 0 \\ \text{where} & \\ & \bar{\mathbf{x}_{ij}} = [\mathbf{x}_{ij}\ \mathbf{r}_{ij}^{i}\ \mathbf{t}_{(i+1)j}^{i}] \end{aligned} \tag{5.3}$$

The augmented Lagrangian method for ATC is used to solve the multi-levels system design problem in the remaining text, and the comprehensive review of the augmented Lagrangian method for ATC is found in Reference [67].

## 5.2 Passive Vehicle Suspension Design in Complex System

Same as Section 4.2, the optimal design of a passive vehicle suspension, shown in Figure 5.3, is studied following Lu *et al.* [60]. The problem only evaluates the tire stiffness, spring stiffness and damping coefficient, in this section, we also design the geometry of the spring and damper to achieve the spring stiffness and damping coefficient to minimize the mean square value of the vertical vibration acceleration of the vehicle body. In this section, the optimization models of all-in-one system of passive vehicle suspension design will be introduced in Section 5.2.1 then the optimization models of multi-level system design will be formulated in Section 5.2.2, and the optimal results would be discussed in Section 5.2.3.

### 5.2.1 All in One System of Passive Vehicle Suspension Design

In this section, we will construct the optimization model including the geometry design of spring and damper in all-in-one system. deterministic design, RBDO, RBDO with inadequate uncertainty data about the passive suspension design

The optimal design of a passive vehicle suspension, shown in Figure 5.3, is studied following Lu *et al.* [60]. The objective is to minimize the mean square value of the vertical vibration acceleration of the vehicle body, which satisfies the following constraints: a lower bound on the road-holding ability of the vehicle ($g_1$); an upper bound on the rolling angle ($g_2$); a lower bound on the suspension's dynamic displacement to avoid bumper hitting, the so-called rattle-space constraint ($g_3$); a lower bound on tire stiffness because tire life is an increasing function of tire stiffness ($g_4$) ; an upper bound on shear stress in the spring bar ($g_5$); a constraint confirm the laminar flow through the orifice ($g_6$); a layout constraint ($g_7$); and an upper bound on admissible orifice diameter. The constraints $g_5$ to $g_8$ are related to the geometry of spring and damper as Figure 5.4 shown.

We study the same problem with different uncertainty level, namely, deterministic problem with no uncertainty, RBDO with uncertainties known for distributions and RBDO problem with inadequate uncertainty data. Following, the optimization models of these three design problem will be constructed.

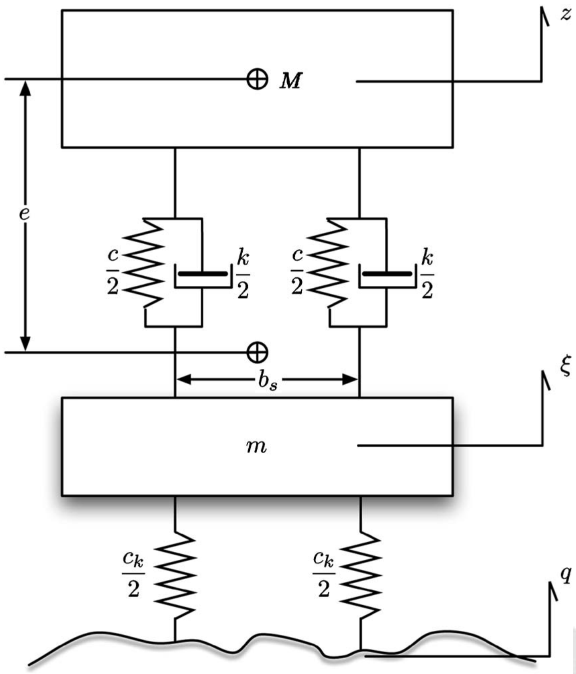



Figure 5.3: Passive vehicle suspension

**Deterministic Optimization Model**

$$
\begin{aligned}
&\min_{c_k,d,D,i_c,d_P,d_S,d_O} \ddot{Z}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2k^{-1}) \\
&g_1 = \left(\frac{\pi AVm}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{M+m} - \frac{c}{M}\right)^2 + \frac{c^2}{Mm} + \frac{c_k k^2}{mM^2}\right) - 1 \le 0 \\
&g_2 = 7.6394(4000(Mg)^{-1.5}c - 1) - 1 \le 0 \\
&g_3 = 0.5(Mg)^{1/2}(k^2 c_k c^{-1}(M+m)^{-1} + c)^{-1/2} - 1 \le 0 \\
&g_4 = ((M+m)g)^{0.877}c_k^{-1} - 1 \le 0 \\
&g_5 = \tau_{\text{spring}} - \tau_{\text{adm}} \le 0 \\
&g_6 = Re - 5000 \le 0 \\
&g_7 = d_P - (D-d) \le 0 \\
&g_8 = d_O - \frac{d_P - d_S}{2} \le 0
\end{aligned}
\tag{5.4}
$$

The problem parameters are provided in Table 5.1.

The spring stiffness $c$ and damping coefficient $k$ can be expressed as function of spring and

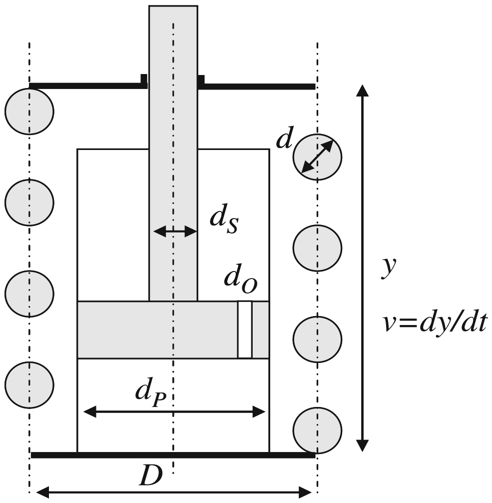


Figure 5.4: Spring and damper

damper geometry as Figure 5.4. The formula is shown in Equation (5.5) to (5.6)

$$c = \frac{d^4 G}{8D^3 i_c} \tag{5.5}$$

$$k = 128\nu \frac{(d_P^2 - d_S^2)/4)^2}{d_O^4 L_t} \tag{5.6}$$

where $d$ is the wire diameter, $D$ is the coil diameter, $i_c$ is the number of coils, $d_P$ and $d_S$ are the diameter of the piston and shaft respectively (see Figure 5.4). The spring stiffness $c$ and damper coefficient should be regarded as two equality constraints in Equation (5.4).

The constraint $g_5$ is an upper bound of shear stress in the spring bar, the shear stress of spring can be expressed as

$$\tau_{\text{spring}} = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2} \tag{5.7}$$

where $F$ is the weight of the vehicle.

The constraint $g_6$ is used to confirm that the oil is maintain the laminar flow through the orifice, and the Reynolds number ($Re$) in $g_6$ can be formulated as

$$Re = \frac{\rho_{\text{oil}} v_{\text{oil}} d_O}{\nu} \tag{5.8}$$

Table 5.1: Suspension problem parameters

| | |
|---|---|
| Dynamic load coefficient, $b_0$ | 0.27 |
| Vehicle velocity, $V$ (m/s) | 10 |
| Gravity acceleration, $g$ (cm/s$^2$) | 981 |
| Road irregularity coefficient, $A$ (cm$^2$ cycle/m) | 1 |
| Sprung mass, $M$ (kg/cm $\cdot$ s$^2$) | 3.2 |
| Unsprung mass, $m$ (kg/cm/s$^2$) | 0.8 |
| Oil velocity, $v_{\text{oil}}$ (cm/s) | 100 |
| Oil dynamic viscosity, $\nu$ (Pas) | 0.16 |
| Oil density, $\rho_{\text{oil}}$ (kg/ m$^3$) | 900 |
| Admissible shear stress, $\tau_{\text{adm}}$ (N/mm$^2$) | 660 |

**RBDO Model**

$$
\begin{aligned}
&\min_{c,c_k,k} \ddot{Z}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2 k^{-1}) \\
&\mathbf{x} = [d, D, i_c, d_P, d_S, d_O, c_k] \\
&g_1 = \Pr\left[\left(\frac{\pi \mathbf{A} V \mathbf{m}}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{\mathbf{M}+\mathbf{m}} - \frac{c}{\mathbf{M}}\right)^2 + \frac{c^2}{\mathbf{Mm}} + \frac{c_k k^2}{\mathbf{mM}^2}\right) - 1 \leq 0\right] \geq R_t \\
&g_2 = \Pr[7.6394(4000(\mathbf{M}g)^{-1.5}c - 1) - 1 \leq 0] \geq R_t \\
&g_3 = \Pr[0.5(\mathbf{M}g)^{1/2}(k^2 c_k c^{-1}(\mathbf{M}+\mathbf{m})^{-1} + c)^{-1/2} - 1 \leq 0] \geq R_t \\
&g_4 = \Pr[((\mathbf{M}+\mathbf{m})g)^{0.877} c_k^{-1} - 1 \leq 0] \geq R_t \\
&g_5 = \Pr[\tau_{\text{spring}} - \tau_{\text{adm}} \leq 0] \geq R_t \\
&g_6 = \Pr[Re - 5000 \leq 0] \geq R_t \\
&g_7 = d_P - (D - d) \leq 0 \\
&g_8 = d_O - \frac{d_P - d_S}{2} \leq 0 \\
&h_1 : k = 128\nu \frac{(d_P^2 - d_S^2)/4)^2}{d_O^4 L_t} \\
&h_2 : c = \frac{d^4 G}{8 D^3 i_c}
\end{aligned}
\tag{5.9}
$$

The constraint $g_1$ to constraint $g_6$ are reliability constraints. The reliability target is given as $R_t = 0.9$. The underlying distributions of uncertain parameters are followed Gaussian distribution as shown in Table 5.2.

Table 5.2: Uncertainty of passive vehicle suspension design in complex system

| | |
|---|---|
| **A** | $\mathbf{A} \sim N(1, 0.03^2)$ cm$^2$/(cycle $\cdot$ m) |
| **M** | $\mathbf{M} \sim N(3.2, 0.03^2)$ Kg $\cdot$ sec$^2$/cm |
| **m** | $\mathbf{m} \sim N(0.8, 0.005^2)$ Kg $\cdot$ sec$^2$/cm |
| $\nu$ | $\nu \sim N(0.16, 0.0016^2)$ Pas |
| $\rho_{\text{oil}}$ | $\rho \sim N(900, 9^2)$ (kg/m$^3$) |

**RBDO Model with Inadequate Uncertainty Data**

When the underlying distributions of uncertainties are unknown, the RBDO problem becomes RBDO with inadequate uncertainty data, then Equation (5.9) can be transformed as Equation (5.10). The initial samples are given in Table 5.3. Table 5.4 shows the relationship between parameters in form of samples and constraints.

Table 5.3: Available initial data of uncertainties in Equation 5.10

| | | |
|---|---|---|
| **A** | 1.00174959561666 | 0.982775981128189 |
| **M** | 3.20505014964935 | 3.24547705458722 |
| **m** | 0.795651191446309 | 0.799194677228595 |
| $\nu$ | 0.160761013223448 | 0.159443262871706 |
| $\rho_{oil}$ | 898.419085163724 | 903.216378191180 |

$$
\begin{aligned}
&\min_{c,c_k,k} \overline{\ddot{Z}}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2 k^{-1}) \\
&\mathbf{x} = [d, D, i_c, d_P, d_S, d_O, c_k] \\
&g_1 = \Pr\left[\Pr\left[\left(\frac{\pi \mathbf{A} V \mathbf{m}}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{\mathbf{M}+\mathbf{m}} - \frac{c}{\mathbf{M}}\right)^2 + \frac{c^2}{\mathbf{Mm}} + \frac{c_k k^2}{\mathbf{mM}^2}\right) - 1 \leq 0\right] \geq R_t\right] \geq CB \\
&g_2 = \Pr[\Pr[7.6394(4000(\mathbf{M}g)^{-1.5}c - 1) - 1 \leq 0] \geq R_t] \geq CB \\
&g_3 = \Pr[\Pr[0.5(\mathbf{M}g)^{1/2}(k^2 c_k c^{-1}(\mathbf{M}+\mathbf{m})^{-1} + c)^{-1/2} - 1 \leq 0] \geq R_t] \geq CB \\
&g_4 = \Pr[\Pr[((\mathbf{M}+\mathbf{m})g)^{0.877} c_k^{-1} - 1 \leq 0] \geq R_t] \geq CB \\
&g_5 = \Pr[\Pr[\tau_{\text{spring}} - \tau_{\text{adm}} \leq 0] \geq R_t] \geq CB \\
&g_6 = \Pr[\Pr[Re - 5000 \leq 0] \geq R_t] \geq CB \\
&g_7 = d_P - (D - d) \leq 0 \\
&g_8 = d_O - \frac{d_P - d_S}{2} \leq 0 \\
&h_1 : k = 128\nu \frac{(d_P^2 - d_S^2)/4)^2}{d_O^4 L_t} \\
&h_2 : c = \frac{d^4 G}{8D^3 i_c}
\end{aligned}
\tag{5.10}
$$

The confidence bound ($CB$) will be updated with the increment of samples. The reliability target is given as $R_t = 0.9$. The confidence range target is given as $CR_t = 0.9$. The constraints' confidence bound limit of optimum must satisfy the confidence range target. Table 5.4 shows the relationship between parameters in form of samples and constraints.

Table 5.4: Parameters respect to the constraints of the passive vehicle suspension design problem

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $\mathbf{A}$ | ✓ | | | | | |
| $\mathbf{M}$ | ✓ | ✓ | ✓ | ✓ | ✓ | |
| $\mathbf{m}$ | ✓ | | ✓ | ✓ | ✓ | |
| $\nu$ | | | | | | ✓ |
| $\rho_{oil}$ | | | | | | ✓ |

### 5.2.2 Multilevel Passive Vehicle Suspension Design

In this section, we will partition the passive vehicle suspension design (Section 5.2.1) into multi-level system design. If the equality constraints can be considered as analysis models, then we assume the $c, k$ are the responses from analysis model $h_1, h_2$. Therefore, the Equation (5.4), (5.9) and (5.10) are partitioned as shown in Figure 5.5. Here the top level is the system-level of passive vehicle suspension design (denote as $P_V$) and the lower level is the subsystem level about the geometry design of damper (denote as $P_D$) and spring (denote as $P_S$). The system-level problem is evaluated the mean square value of the vertical vibration acceleration of the vehicle body. The subsystem level design problem calculate the response from each corresponding analysis model $h_1$ and $h_2$, and they share the linking variables. Input of system-level design problem is tire stiffness ($c_k$), while the responses from lower level system are damping coefficient ($k$) and spring stiffness ($c$). Inputs of subsystem level design problems are the number of coils $i_c$, the diameter of the piston $d_P$ and the diameter of the shaft $d_S$, the responses $c$, $k$ returned from the geometry of spring and damper, the linking variables between the subsystem level design problem are wire diameter $d$ and coil diameter $D$. Categorization of responses and variables is given in Table 5.5. Only the value of $c, k, d, D$ are passed up and cascaded down between the system and subsystem level design problems.

Following, we only construct the multi-level optimization model of RBDO with inadequate uncertainty data, the deterministic optimization model and RBDO model can be constructed by analogy.

Table 5.5: Summary of responses and variables of passive vehicle suspension design

| | Responses | Local variables | Linking variables | Responses from lower level | Uncertainties |
|---|---|---|---|---|---|
| Suspension System | N/A | $c_k$ | N/A | $c, k$ | $\mathbf{A}, \mathbf{M}, \mathbf{m}$ |
| Damper | $k$ | $d_P, d_S, d_O$ | $d, D$ | N/A | $\nu, \rho_{\text{oil}}$ |
| Spring | $c$ | $i_c$ | $d, D$ | N/A | $\mathbf{M}, \mathbf{m}$ |

The system-level design problem (passive vehicle suspension design problem) can be ex-

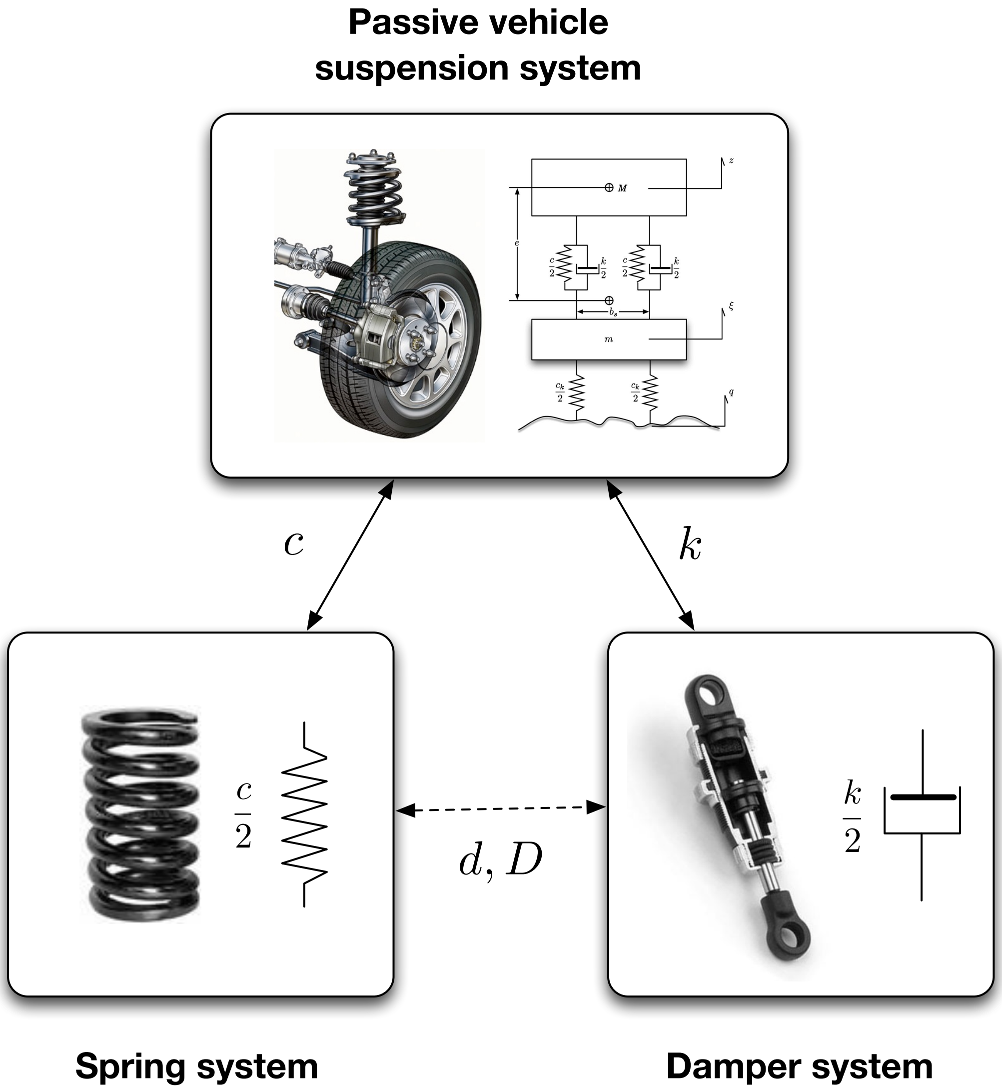



Figure 5.5: Multi-level passive vehicle suspension design structure

pressed as Equation (5.11).

$$
\begin{aligned}
&\text{System-level : Passive vehicle suspension design problem } P_V \\
&\min_{c,c_k,k,d,D} \ddot{Z}^2 = (\pi AV/m^2)(c_k k + (M+m)c^2 k^{-1}) + \pi_{AL}(c,k,d,D) \\
&g_1 = \Pr\left[\Pr\left[\left(\frac{\pi \mathbf{A} V \mathbf{m}}{b_0 g^2 k}\right)\left(\left(\frac{c_k}{\mathbf{M}+\mathbf{m}} - \frac{c}{\mathbf{M}}\right)^2 + \frac{c^2}{\mathbf{Mm}} + \frac{c_k k^2}{\mathbf{mM}^2}\right) - 1 \le 0\right] \ge R_t\right] \ge CB \\
&g_2 = \Pr[\Pr[7.6394(4000(\mathbf{M}g)^{-1.5}c - 1) - 1 \le 0] \ge R_t] \ge CB \\
&g_3 = \Pr[\Pr[0.5(\mathbf{M}g)^{1/2}(k^2 c_k c^{-1}(\mathbf{M}+\mathbf{m})^{-1} + c)^{-1/2} - 1 \le 0] \ge R_t] \ge CB \\
&g_4 = \Pr[\Pr[((\mathbf{M}+\mathbf{m})g)^{0.877} c_k^{-1} - 1 \le 0] \ge R_t] \ge CB
\end{aligned}
\tag{5.11}
$$

The constraints $g_1$ to $g_4$ are constrained to achieve a certain confidence range to provide a creditable reliability estimation. The reliability target is given as $R_t = 0.9$ and the confidence range target is also given as $CR_t = 0.9$. The confidence bound ($CB$) will be updated with the increment of samples. The constraints' confidence bound limit of optimum must satisfy the confidence range target.

The two subsystem (damper system and spring system) design problem can be stated as Equation (5.12) and (5.13). For each problem, the design objective is to minimize the deviations between the targets and responses or linking variables. The damper design problem gets the response $k$ from analysis model $h_1$, and the spring design problem gets the response $c$ from analysis model $h_2$. The variables $d$ and $D$ are subsystem linking variables. The parameters $k^U, c^U, d^U, D^U$ are target values cascaded down from passive vehicle suspension design problem. The constraints $g_5, g_6$ are constrained to achieve a certain confidence range to provide a creditable reliability estimation. The reliability target is given as $R_t = 0.9$ and the confidence range target is also given as $CR_t = 0.9$. The confidence bound ($CB$) will be updated with the increment of samples. The constraints' confidence bound limit of optimum must satisfy the confidence range target.

$$
\begin{aligned}
&\text{Subsystem level : Damper design problem } P_D \\
&\min_{k,d_P,d_S,d_O,d,D} (k-k^U)^2+(d-d^U)^2+(D-D^U)^2 \\
&g_6 = \Pr\left[\Pr\left[Re-5000 \le 0\right] \ge R_t\right] \ge CB \\
&g_7 = d_P-(D-d) \le 0 \\
&g_8 = d_O - \frac{d_P-d_S}{2} \le 0 \\
&h_1 : k = 128\nu \frac{(d_P^2-d_S^2)/4)^2}{d_O^4 L_t}
\end{aligned}
\tag{5.12}
$$

where

$$
Re = \frac{\rho_{\text{oil}} v_{\text{oil}} d_O}{\nu}
$$

$$
\begin{aligned}
&\text{Subsystem level : Spring design problem } P_S \\
&\min_{c,i_c,d,D} (c-c^U)^2+(d-d^U)^2+(D-D^U)^2 \\
&g_5 = \Pr\left[\Pr\left[\tau_{\text{spring}}-\tau_{\text{adm}} \le 0\right] \ge R_t\right] \ge CB \\
&h_2 : c = \frac{d^4 G}{8D^3 i_c}
\end{aligned}
\tag{5.13}
$$

where

$$
\tau_{\text{spring}} = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}
$$

### 5.2.3 Optimal Results and Discussion

In this section, the optimal results of all-in-one system design and multi-level system design would be given and discussed. We focus on the results of sampling augmentation application of multi-level system. First, the optimal results will be compared between all-in-one system design and multi-level system design, then the detailed results of all-in-one system would be demonstrated, and then the optimal results of multi-level system design would be shown. The results of deterministic design, RBDO, and RBDO with inadequate uncertainty data would be compared. We will use Monte Carlo Simulation to acquire the reliability value of the optimal points, which referred as MCS reliability denoted as $R_{MCS}$) to represent as true reliability. About the RBDO with inadequate uncertainty data, the Bayesian reliability $R_B$ defined in section 3.1.2 is used to represent the estimation value of the reliability distribution.

In section 2.3.3, two types of MCMC filter mechanisms are proposed. The difference between these two filters is that when a new sample is rejected, one replicate the previous sample as a new one which the other one take a completely new measurement sample. Therefore, we have three scenarios in our study. Scenario 1 is sampling augmentation with MCMC of accepting current sample which means that when rejected sample occurs, the current sample be accepted as an additional sample. Scenario 2 is sampling augmentation with MCMC of accepting an additional sample which means that when rejected sample occurs, the filter mechanism will continue still an acceptable samples appear. Scenario 3 is sampling augmentation without MCMC filter mechanism. In the following, these three scenarios would be compared to show the effectiveness of MCMC filter.

**Comparison of Optimal Results between All-in-One System and Multi-Level System**

Table 5.6 shows the optimization results obtained from the all-in-one and the analytical target cascading models. And the optimization results with different level of uncertainties are also given. The ATC solutions were obtained following top-down implementation fashion. After solving the system level, and the subsystem problem were cascaded based on the optimal design at the system level, and the subsystem optimal designs were passed back to the system level after minimizing deviations between the responses and targets. This top-down and bottom-up process completed one iteration loop in the ATC process and it was repeated until convergence.

The objective function values of deterministic and RBDO in all-in-one system and multi-level system are quite close, respectively as shown in Table 5.6. But the objective function values of three types sampling augmentation process are different in all-in-one system and multi-level system. The insistency of sample information in multi-level system would affect the search of optimum. Although the optimum of multi-level system is different as all-in-one system, it still give an acceptable optimal design. As the overall results of $R_{MCS}$, the all-in-one system provide a higher reliability results, and higher confidence range about the reliability distribution which is larger than the reliability target.

Table 5.6: Comparison all in one and ATC optimal result with and without MCMC

| | | $\ddot{Z}^2$ | $R_{MCS}$ | Confidence range |
|---|---|---|---|---|
| | Deterministic | 2819049 | (1,0.496,0.498,0.499,1,1) | N/A |
| | RBDO | 2881346 | (1,0.90,0.90,0.90,1,1) | N/A |
| All-in-one | Scenario 1 | 2927656 | (1,0.9404,1,0.9365,1,1) | (1,1,1,1,1,1) |
| | Scenario 2 | 2899148 | (1,0.9404,0.9385,0.9808,1,1) | (1,1,1,1,1,1) |
| | Scenario 3 | 2899148 | (1,0.9404,0.9385,0.9808,1,1) | (1,1,1,1,1,1) |
| | Deterministic | 2819019 | (1,0.496,0.498,0.498,1,1) | N/A |
| | RBDO | 2887505 | (1,0.90,1,0.90,1,1) | N/A |
| Multi-level | Scenario 1 | 2888959 | (0.9999,0.9215,0.9223,0.9468,1,1) | (1,0.940,1,1,1,1) |
| | Scenario 2 | 2976893 | (0.9496,0.9215,1,1,1,1) | (0.9496,0.9215,1,1,1,1) |
| | Scenario 3 | 3051018 | (0.70,0.9215,1,1,1,1) | (0.9496,0.9215,1,1,1,1) |

**Optimal Results in All-in-One System**

Table 5.7 shows the comparison of deterministic, RBDO ,RBDO with inadequate uncertainty data (MCMC of accepting current sample). Deterministic design is assumed that there are no uncertainties, in other words, we assume that we exactly know the parameter values. The optimum of deterministic optimization should be more affirmatory than which with uncertainties. RBDO with inadequate uncertainty data should be the most conservative one of these three types optimization problems. Due to lack of information about the uncertainties, the reliability estimation in RBDO with inadequate uncertainty data is also regarded as a uncertain quantity. Therefore, the reliability becomes a distribution and the reliability estimation should confirm a certain degree confidence range target (the concept is the same as the reliability value of this reliability distribution). In order to confirm the confidence range of reliability distribution, the optimum becomes most conservative one of these three types optimization problems. Although RBDO with inadequate uncertainty data is the most conservative one, it still give an acceptable optimal results.

In the practical engineering community, the characteristics of underlying distribution about the uncertainty cannot be actually known, what we can do is measuring sampling from the underlying distribution and make inference about the underlying distribution. Optimal sampling

Table 5.7: Comparison of optimal results in all-in-one problem

| | AiO With MCMC(current sample) | AiO RBDO | Aio Deterministic |
|---|---|---|---|
| $d$ (mm) | 5.3256 | 5.112 | 4.972 |
| $D$ (mm) | 120.4 | 116.0 | 112.5 |
| $i_c$ | 12 | 12 | 12 |
| $d_P$ (mm) | 20.17 | 20.58 | 15.97 |
| $d_S$ (mm) | 10.58 | 10.60 | 10.00 |
| $d_O$ (mm) | 3.907 | 4.143 | 2.932 |
| $c$ | 388.0 | 446.8 | 401.3 |
| $c_k$ | 1440 | 1438 | 1426 |
| $k$ | 23.90 | 20.77 | 20.79 |
| $\ddot{\bar{Z}}^2$ | 2927656 | 2881346 | 2819049 |
| $R_{MCS}$ | (1,0.9404,1,0.9365,1,1) | (1,0.90,0.90,0.90,1,1) | (1,0.496,0.498,0.499,1,1) |
| Confidence range | (1,1,1,1,1,1) | N/A | N/A |

augmentation provide an acceptable estimation of reliability distribution via measuring additional samples during optimization iterations. And we can save a lot of fund to sampling great amount of samples to represent the underlying distribution of uncertainty .

Table 5.8 shows the comparison of three types sampling augmentation processes. Three sampling augmentation processes give additional measurements on parameter $\mathbf{M}$ relative to the critical constraint $g_4$. The filter mechanism MCMC would help the convergency of optimization in sampling augmentation process. Scenario 3 need to cast 43 additional samples to obtain a feasible solution. Yet, both two sampling augmentation with MCMC (Scenario 2 and 3) cast fewer samples to give acceptable solutions. Because the biased samples would make the reliability estimation unstable, and the confidence range of reliability distribution would fluctuate up and down, then the search directions of optimization would be affected by fluctuated confidence range values.

Although scenario 2 give a better optimum than which scenario 1 given, it use more samples actually (20+14 v.s. 20). Scenario 1 replicate the previous sample as a new one when a new sample is rejected. In the view of limit resource of measurement, we can assert that the

Table 5.8: Comparison of optimal results in all-in-one problem with filter mechanism

| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $d$ (mm) | 5.3256 | 5.177 | 5.177 |
| $D$ (mm) | 120.4 | 117.4 | 117.4 |
| $i_c$ | 12 | 12 | 12 |
| $d_P$ (mm) | 20.17 | 20.04 | 20.04 |
| $d_S$ (mm) | 10.58 | 10.57 | 10.57 |
| $d_O$ (mm) | 3.907 | 3.999 | 3.999 |
| $c$ | 388.0 | 388.0 | 388.0 |
| $c_k$ | 1440 | 1446 | 1446 |
| $k$ | 23.90 | 21.04 | 21.04 |
| $\ddot{Z}^2$ | 2927656 | 2899148 | 2899148 |
| $R_B$ | (0.99,0.99,0.99,0.99,0.99,0.99) | (0.99,0.94,0.91,0.96,0.99,0.99) | (1,0.92,0.91,0.96,1,1) |
| $R_{MCS}$ | (1,0.94,1,0.94,1,1) | (1,0.94,0.94,0.98,1,1) | (1,0.94,0.94,0.98,1,1) |
| Confidence range | (1,1,1,1,1,1) | (1,1,0.98,1,1,1) | (1,1,0.98,1,1,1) |
| Adding procedure | 20 on $\mathbf{M}$ | 20 on $\mathbf{M}$ | 43 on $\mathbf{M}$ |
| No.rejected sample | 7 | 14 | N/A |
| No. combination | 352 | 352 | 720 |
| Active constraint | $g_4$ | $g_4$ | $g_4$ |

sampling augmentation with MCMC of accepting current sample is the best one choice when we want to use the least samples.

### Optimization Results in Multi-Level System

Table 5.9 shows the comparison of deterministic, RBDO ,RBDO with inadequate uncertainty data (MCMC of accepting current sample). In intuitive, the less amount of information, the more conservative optimal result obtained. The optimal results of RBDO with inadequate uncertainty data should be the most conservative one. The function value $\ddot{Z}^2$ shows that the RBDO with inadequate uncertainty data is the most conservative one. We know that reliability estimation in RBDO with inadequate uncertainty data could be regarded as a uncertain

Table 5.9: Comparison of optimal results of passive vehicle suspension design in complex system

| | Mutli-level With MCMC(current sample) | Mutli-level RBDO | Mutli-level Deterministic |
|---|---|---|---|
| $d$ (mm) | 8.894 | 8.890 | 8.851 |
| $D$ (mm) | 200 | 200 | 200 |
| $i_c$ | 20 | 20 | 20 |
| $d_P$ (mm) | 22.50 | 20.00 | 19.64 |
| $d_S$ (mm) | 10.00 | 12.16 | 10.29 |
| $d_O$ (mm) | 1.00 | 3.691 | 3.941 |
| $c$ | 387.2 | 386.6 | 379.8 |
| $c_k$ | 1441 | 1438 | 1426 |
| $k$ | 21.04 | 21.96 | 20.77 |
| $\ddot{Z}^2$ | 2888959 | 2887505 | 2819019 |
| $R_{MCS}$ | (0.9999,0.9215,0.9223,0.9468,1,1) | (1,0.90,1,0.90,1,1) | (1,0.496,0.498,0.498,1,1) |
| Confidence range | (1,0.940,1,1,1,1) | N/A | N/A |

quantity as previous assertion. The reliability should achieve a certain confidence level the satisfying the probability of the reliability distribution larger than the reliability target should be bigger than the confidence range target. Therefore, in order to comply with the confidence range target, the optimum becomes more conservative than which of RBDO obtained. Yet, the sampling augmentation is given a quite closer optimum comparing to which RBDO given. Sampling augmentation cast additional samples to increase the accuracy of reliability estimation during optimization iterations. Therefore, we can save a lot of fund to sampling great amount of samples to represent the underlying distribution of uncertainty ( we use $10^6$ samples to simulate the underlying distribution ) by using the sampling augmentation with MCMC.

The comparison of optimal result with filter mechanisms is provided in Table 5.10. Three sampling augmentation processes give additional measurements on parameter $\mathbf{M}$ relative to the critical constraint $g_4$. Both sampling augmentations with MCMC perform better on function value $\ddot{Z}^2$ than on which without MCMC. The geometry design of spring have no significant difference. The most difference is the design of tire stiffness $c_k$. The convergency rate would be improved by MCMC filter mechanism. As Table 5.10 shown,scenario 3 uses 50 additional

Table 5.10: Comparison of optimal results in complex system with filter mechanism

| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $d$ (mm) | 8.894 | 8.894 | 8.894 |
| $D$ (mm) | 200 | 200 | 200 |
| $i_c$ | 20 | 20 | 20 |
| $d_P$ (mm) | 22.50 | 22.66 | 22.96 |
| $d_S$ (mm) | 10.00 | 10.00 | 10.00 |
| $d_O$ (mm) | 1.00 | 1.00 | 1.00 |
| $c$ | 387.2 | 386.6 | 379.8 |
| $c_k$ | 1441 | 1518 | 1569 |
| $k$ | 21.04 | 21.85 | 22.92 |
| $\ddot{\bar{Z}}^2$ | 2888959 | 2976893 | 3051018 |
| $R_B$ | (1,0.91,1,1,1,1) | (1,0.91,1,1,1,1) | (0.99,0.99,1,1,1,1) |
| $R_{MCS}$ | (0.9999,0.9215,0.9223,0.9468,1,1) | (0.9496,0.9215,1,1,1,1) | (0.70,0.9215,1,1,1,1) |
| Confidence range | (1,0.99,1,1,1,1) | (1,0.99,1,1,1,1) | (0.99,0.99,1,1,1,1) |
| Adding procedure | 40 on **M** | 40 on **M** | 50 on **M** |
| No.rejected sample | 20 | 14 | N/A |
| No. combination | 672 | 672 | 848 |
| Active constraint | $g_4$ | $g_4$ | $g_4$ |

samples (and 50 optimization iterations) to obtain the feasible solution and the reliability is poor comparing with optimum with MCMC. Because the biased samples would affect the reliability inference and then the confidence range of reliability distribution becomes unstable and fluctuated. The convergent rate may be slow down when the inconsistency of constraint value (confidence range) occurred. By the filter mechanism, the effect of biased samples can be eliminated, then the convergent rate can be improved.

Both sampling augmentation with MCMC use same number of samples to make inference in optimization models, however, scenario 2 casts more additional samples. Actually, the filter mechanism MCMC of accepting an additional samples measures additional samples until an acceptable samples occur when the rejected sample exists. So the number of additional samples of scenario 2 is $40+14 = 54$, which is more than scenario 1 (40). Scenario 1 spend less additional samples to perform optimization, it gives a set of optimal point with smaller function value and higher $R_B$ than which with MCMC of accepting an additional sample. Therefore, we can assert that sampling augmentation with MCMC of accepting current sample could use the least samples to provide a creditable reliability estimation and optimal result.

## 5.3 Summary

In this chapter, we extend the sampling augmentation to multi-level system design with inadequate uncertainty data. In engineering community, the multi-level systems design is necessary. No single design group could handle a complex system design problem. If there are also inadequate uncertainty data in multi-level system design problem, the sampling augmentation would also be helpful to use fewer samples to estimate reliability and allocate the resource efficiently. We see all constraints in different system level design problem to decide which constraint is critical one which is same as the all-in-one system. Because the measurement of samples is expensive, the efficient resource allocation is prerequisite. We only measure additional samples when the corresponding uncertain parameter is important.

However, allocating resource efficiently is not enough, measurements sometimes will go wrong, the filter mechanism MCMC is used to avoid higher biased samples. Both two sampling augmentations with MCMC filter give a significant effect on filtering biased samples and improvement of convergency of optimization. We use the concept of sample combination instead of a set of all uncertainties to do Bayesian binomial inference. The Bayesian reliability $R_B$ is quite close to the MCS reliability $R_{MCS}$, therefore, we can assert that the concept of sample combination helps us to reveal all possible situations that each sample be matched and give a creditable reliability estimation. In this chapter, we can see that the sampling augmentation could also be applied to multi-level system design and have a good performance on reliability estimation and resource allocation.

# Chapter 6 Conclusions and Future Work

## 6.1 Conclusions

The goal of this thesis is to *provide a systematic approach for sample augmentation such that the final reliability estimation of a complex system can be acceptable to a specific level.*

In Chapter 2, the evaluation of reliability with inadequate uncertainty data is done via Bayesian inference. An MCMC sample data filter is also developed to avoid the influence of biased samples. In Chapter 3, an optimal sampling augmentation in RBDO with inadequate uncertainty data is proposed to cast additional samples with least resources. The addition of samples is decided based on the allocation of resources and based on the quality of the samples and the biased samples are filtered via MCMC. In Chapter 4, two case studies in single level system are demonstrated to show the validity of the proposed method. In Chapter 5, the proposed method is extended to multilevel system to reveal the real complexity of engineering design.

The specific contributions of this thesis are summarized into four main points :

1. **This thesis uses limited samples to give an acceptable optimal result and accurate reliability estimation :** The measurement of samples can be costly. In literature, the amount of samples used to perform reliability analysis is still too large. In this thesis, we proposed an optimal sampling augmentation and resource allocation for engineering design. The concept of sample combinations not only reveal all possible situations of samples but also decrease the amount of samples requires to infer a reliability distribution.

2. **This thesis allocates resource more efficiently :** The differences of uncertainties can only be reveal with the concept of sample combinations. We can cast additional samples only when they are important instead of measuring all uncertainties, therefore, the unnecessary redundant measurements can be avoided. The samples on constraints with lowest confidence range and uncertainties with high sensitivity are added. With this strategy we could use the least resource to reach a desired reliability goal.

3. **This thesis filters biased samples via Markov chain Monte Carlo method :** Badly measured samples will affect the accuracy of reliability estimation. The biased samples will also influence the convergency of optimization due to the unstable of confidence range which corresponded to constraint. The proposed provides a a mechanism to avoid the influence of such biased samples. With the MCM filter, advert effects of biased samples can be avoided such that one does not need to cast much more samples to alleviate the influence of biased samples. In addition, the reliability estimation and the optimal sampling augmentation become more robust.

4. **This thesis extents the proposed method to multilevel systems :** Modern engineering products are complex systems design. Designer of these products are separated into several groups to handle different part of design. Inadequate uncertainty data also exist in this architecture. The proposed sampling augmentation method can assist designers to use limited samples to give an acceptable yet reliability optimal solution.

## 6.2 Future Work

The following research activities deserve much in-depth investigation in the future :

1. Take the cost of measurement into consideration. Different uncertainties may have different measurement cost. Under the cost limit, the number of samples might be constrained and the resource allocation strategy should be modified.

2. Considerate the dependency of uncertainties. The sensitivity analysis in resource allocation is assumed that the uncertainties are independent. Unfortunately uncertainties sometimes influence each others. The sensitivity analysis of these dependent uncertainties can be taken into consideration.

3. Deal with design problem with inadequate uncertainty data in objective function. In this thesis, when objective function is also with inadequate uncertainty data, we use the average value of samples data then make it become a deterministic objective function. The probability value of objective function with inadequate uncertainty data need to be discussed.

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